

# Tannaka Duality: Reconstructing Groups from Representations

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October 2024

- 1 What is a Representation?
- 2 Limits of Reconstruction
- 3 More Structure and the Main Result

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2 Limits of Reconstruction

3 More Structure and the Main Result

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- Is there a way to turn group theory into linear algebra?

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## Remark

These objects and homomorphisms form a category

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Both have the same character table  $\chi_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$

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- Is there a way to salvage a positive result from question 4 though?

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## Remark

These give the category of representations the structure of a Tannakian Category

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## Theorem

Given a symmetric monoidal category of representations,  $\text{Rep}(G)$ , and its forgetful functor,  $\omega$ , we can reconstruct  $G$  as  $\underline{\text{Aut}}^{\otimes}(\omega)$

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## Examples

An automorphism of  $\omega$  at a representation  $X$ , on a vector space  $V$  looks like the following

diagram 
$$\begin{array}{ccc} V & \xrightarrow{\eta_X} & V \\ \downarrow \omega(f) & & \downarrow \omega(f) \\ V & \xrightarrow{\eta_X} & V \end{array}$$
 This just says that an automorphism of  $\omega$  is a linear map that commutes with all  $G$  equivariant endomorphisms

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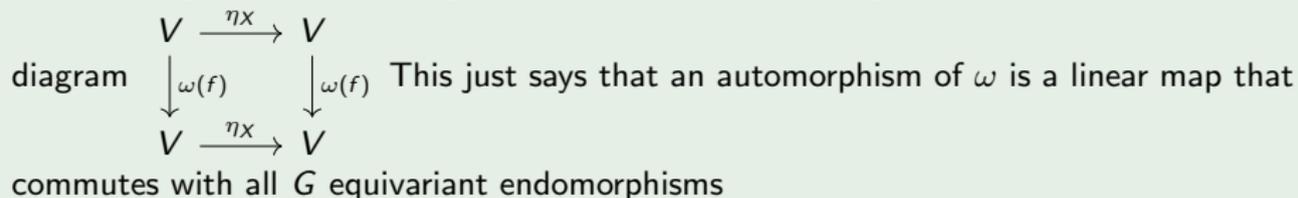
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- Why is this realistic?
- Because by definition of  $G$  equivariant, the action of  $G$  satisfies it