Tannaka Duality: Reconstructing Groups from Representations

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October 2024

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- Group Theory is Hard

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- Linear Algebra is Easy

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- Is there a way to turn group theory into linear algebra?

- Representations allow us to turn group theory into linear algebra

Definition

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Remark

These objects and homomorphisms form a category

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Both have the same character table $\chi_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$

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- Is there a way to salvage a positive result from question 4 though?

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Remark

These give the category of representations the structure of a Tannakian Category

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Main Theorem

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Theorem

Given a symmetric monoidal category of representations, Rep(G), and its forgetful functor, ω, we can reconstruct G as $\underline{\mathrm{Aut}}^{\otimes}(\omega)$

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How the construction works

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Examples

An automorphism of ω at a representation X, on a vector space V looks like the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\eta_X} & V \\ \downarrow{\scriptstyle \omega(f)} & & \downarrow{\scriptstyle \omega(f)} \\ V & \xrightarrow{\eta_X} & V \end{array}$$

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- Why is this realistic?
- Because by definition of G equivariant, the action of G satisfies it