Measuring the Butterfly Velocity in the XY Model on Emerging Quantum Computers

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Summer Research in Maths Programme 2024
Centre for Mathematical Sciences
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October 14th 2024

Introduction

What is a Quantum Computer?

Quantum Computer

A device that maps a sequence of gates to real numbers according to some rules of quantum mechanics (L2 normalised states measured using Born Rule). This allows n qubits to utilise an N dimensional complex Hilbert space for computations.

Qubits

Quantum bits that take the form $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = (\alpha, \beta)^T \in \mathcal{H}$.

Gates

Functions $G : \mathcal{H} \to \mathcal{H}$, generally represented by unitary matrices.

Introduction

XY Model

Discrete spin system defined by Hamiltonian:

$$H = J \sum_j \left(\frac{1+r}{2} X_j X_{j+1} + \frac{1-r}{2} Y_j Y_{j+1} + h Z_j \right).$$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Butterfly Velocity

Speed of information propagation through spin system.

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Out-of-Time-Order Correlator (OTOC)

$$F(t) \equiv \langle W(t)^\dagger V^\dagger W(t) V \rangle_\rho \equiv \mathrm{Tr}(\rho W(t)^\dagger V^\dagger W(t) V)$$

$$W(t) \equiv e^{iHt} W e^{-iHt} \equiv U^\dagger W U$$

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Study and understand information transport in spin systems
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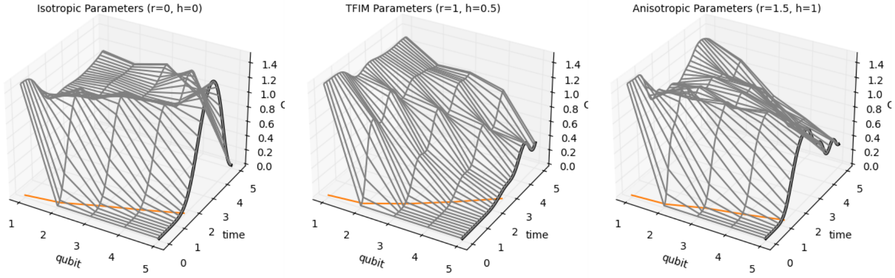
Approach

Build a robust, near-optimal, scalable numerical quantum algorithms for measuring information transport using out-of-time-order correlation functions (OTOCs).

Numerical Results

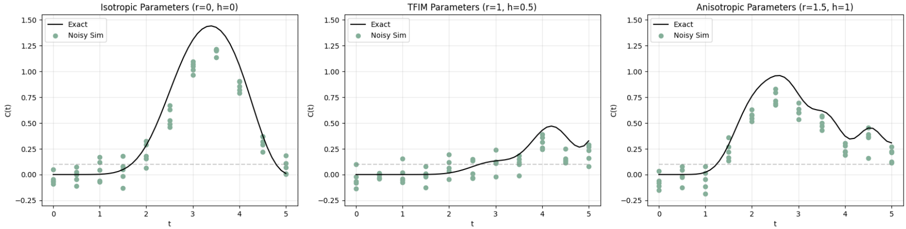
OTOC Wireframe Plots



$$C(t) \equiv \mathrm{Tr}\left(\rho |[W(t), V]|^2\right) = 2 - 2\,\mathrm{Re}\,F(t)$$

Numerical Results

IBM-FakeTorino Noisy Quantum Simulator Plots



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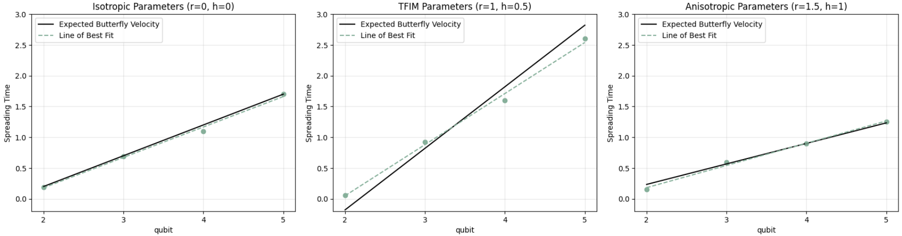
Numerical Results

IBM-Q Simulated Butterfly Velocity



Spreading Time: $t_j = \min_t \{ C_j(t) > 0.1 \}$

YKY Algorithm

$$F(t) = \sum_{V,W} \frac{1}{2^n} \mathrm{Tr}(W(t)VW(t)V)$$

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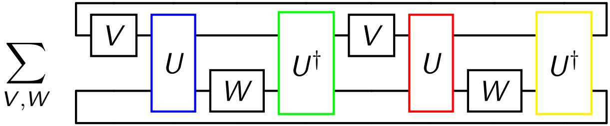

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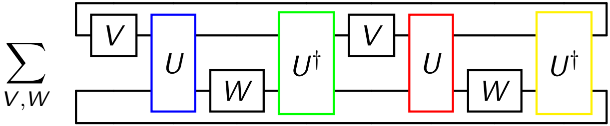

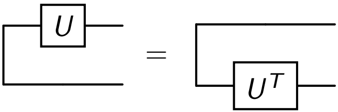

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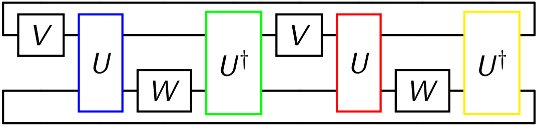
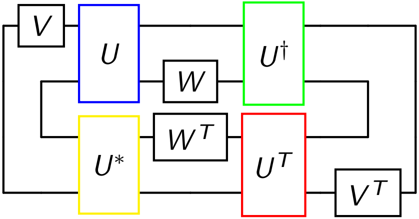
$\sum_{V,W}$ 

$\sum_{V,W}$ 

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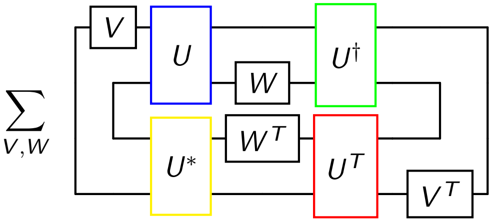
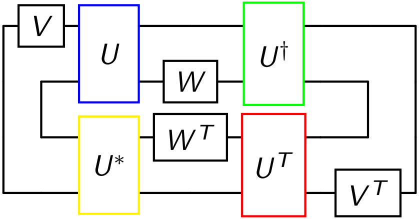
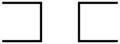


YKY Algorithm

$$\sum_{V,W}$$



$$\sum_{P \in \{I,X,Y,Z\}} P^T \otimes P =$$



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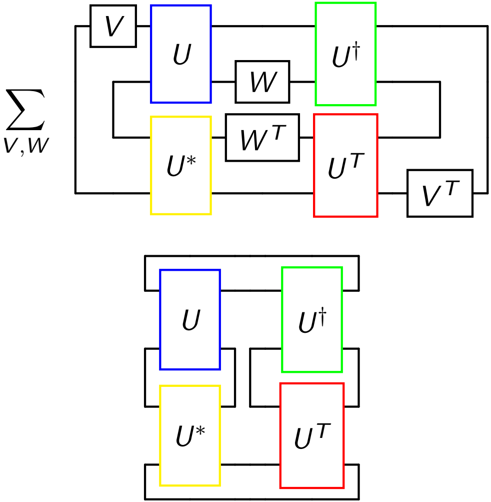

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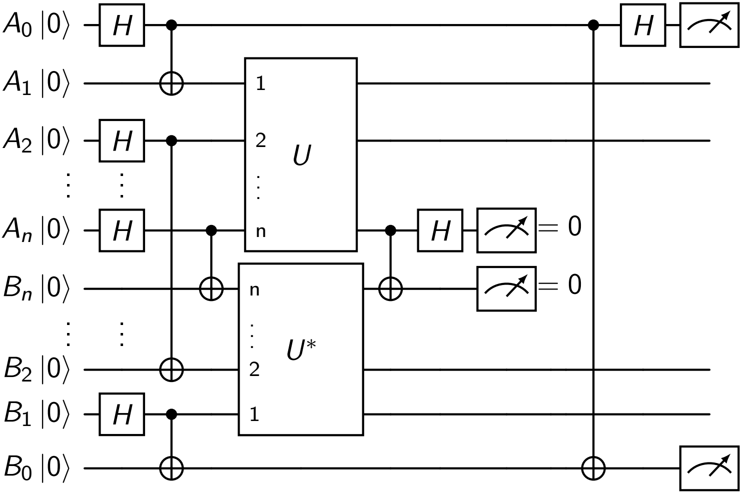

Riemannian Trust Regions

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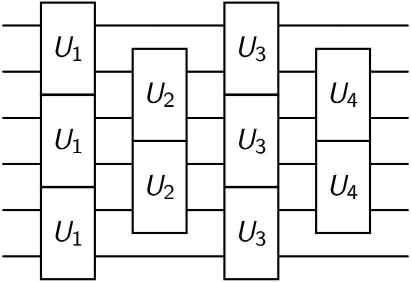

Manifold optimisation method of Riemannian Trust Regions method to map the Hamiltonian H to a set of local $SU(4)$ gates.

Utilise properties of the Hamiltonian, such as translation-invariance.

Non-asymptotically more promising than Product-Splitting methods (such as Lie-Trotter-Suzuki-Strang-Yoshida)

Analytical Calculation

Spin Lattice Hamiltonian for XY Model

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Fermionic Hamiltonian for XY Model

$$H = J\sum_j r(f_{j+1}f_j + f_j^\dagger f_{j+1}^\dagger) + (f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1})$$
$$+ h(\mathbb{I} - 2f_j^\dagger f_j)$$

$$f_j = \underbrace{Z \otimes \ldots \otimes Z}_{j-1} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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Fermionic Momentum Space Hamiltonian for XY Model

$$H = -J\sum_k 2(h - \cos(k))c_k^\dagger c_k + ir\sin(k)(c_{-k}^\dagger c_k^\dagger + c_{-k}c_k) - h\mathbb{I}$$

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Bogoliubov Transform

$$\gamma_k = u_k c_k - iv_k c_{-k}^\dagger$$

$$u_k, v_k \in \mathbb{R}, \quad u_k^2 + v_k^2 = 1, \quad u_{-k} = u_k, \quad v_{-k} = -v_k$$

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Bogoliubov Angle

$$u_k = \cos(\theta_k/2) \qquad v_k = \sin(\theta_k/2)$$

Analytical Calculation

Diagonalised Hamiltonian for XY Model

$$H = \sum_k \varepsilon(k; r, h) \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) \quad (1)$$

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$$H = \sum_k \varepsilon(k; r, h) \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) \tag{1}$$

Energy Dispersion Relation

$$\varepsilon(k; r, h) = -2J\sqrt{(h - \cos k)^2 + r^2 \sin^2 k} \tag{2}$$

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Group Velocity

$$v_g(k; r, h) = -2J \frac{\sin k(h - \cos k) + r^2 \sin k \cos k}{\sqrt{(h - \cos k)^2 + r^2 \sin^2 k}} \tag{3}$$

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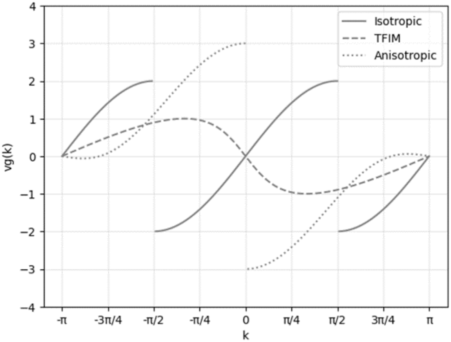

	J	r	h	v_B
Isotropic	1	0	0	2
TFIM	1	1	0.5	1
Anisotropic	1	1.5	1	3

Summary & Future Ideas

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Also, read the pre-print (if Section II ever gets finished...) at arXiv 2410.XXXXX