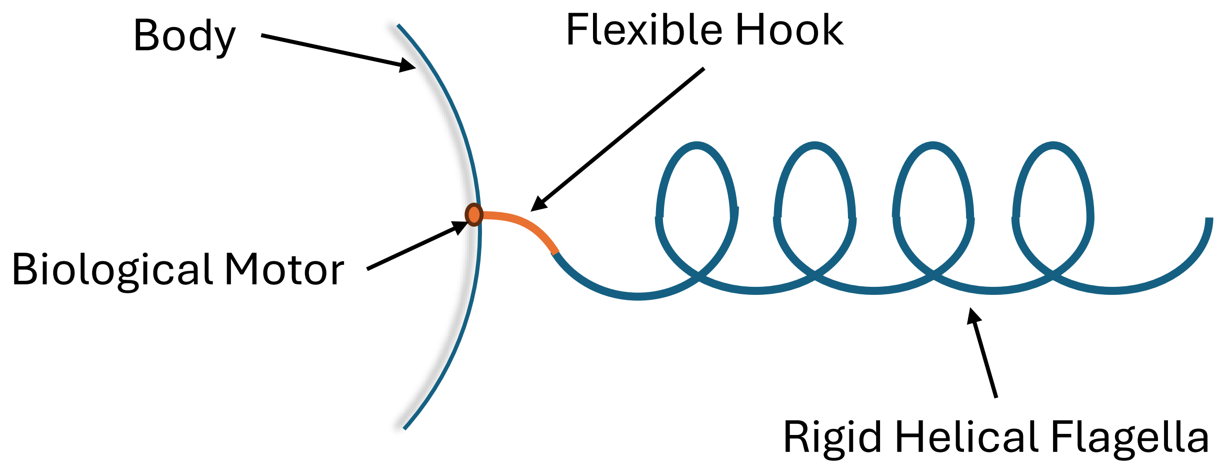
Modelling the stabilities and trajectories of uniflagellar bacteria

Jay Manson-Whitton

Supervisor: John Lister

Introduction



- Rotating rigid helical flagella for propulsion
- Periodic reorientation
- Different reorientation techniques:
  - ‘Run-and-Tumble’
  - ‘Forward-Reverse-Flick’

Bifurcation Overview

- Dynamical System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda)$
- $\lambda = \lambda_0$ is a bifurcation point if the topological structure of the dynamical system changes as λ passes through λ_0

Bifurcation Overview

- Dynamical System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda)$
- $\lambda = \lambda_0$ is a bifurcation point if the topological structure of the dynamical system changes as λ passes through λ_0

- Example (Supercritical Pitchfork):

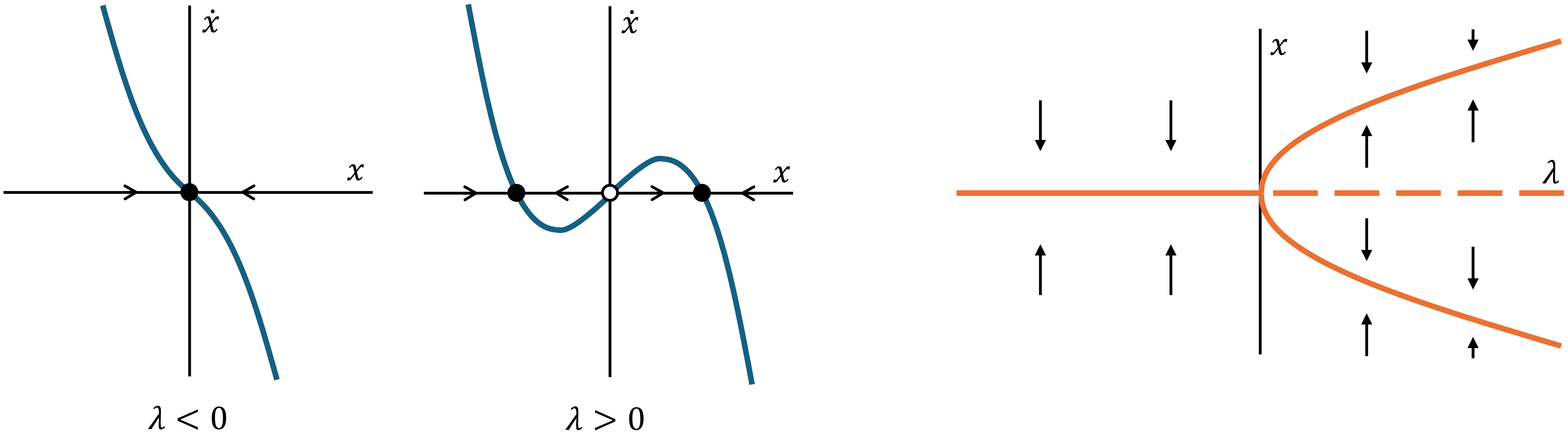
$$\dot{x} = \lambda x - x^3$$

Bifurcation Overview

- Dynamical System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda)$
- $\lambda = \lambda_0$ is a bifurcation point if the topological structure of the dynamical system changes as λ passes through λ_0

- Example (Supercritical Pitchfork):

$$\dot{x} = \lambda x - x^3$$



Bifurcation Overview

- Dynamical System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda)$
- $\lambda = \lambda_0$ is a bifurcation point if the topological structure of the dynamical system changes as λ passes through λ_0

- Example (Subcritical Pitchfork):

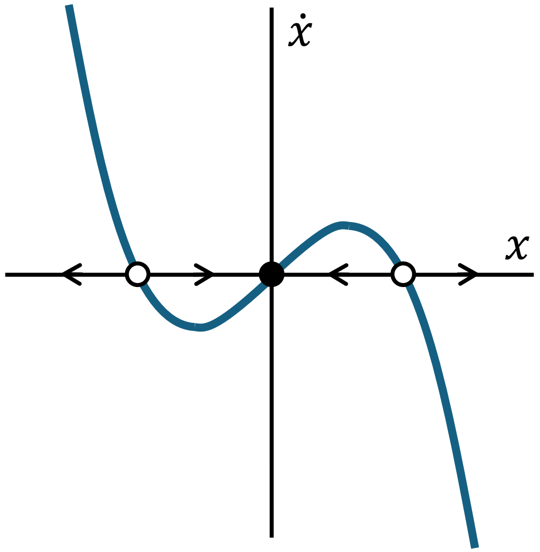
$$\dot{x} = \lambda x + x^3$$

Bifurcation Overview

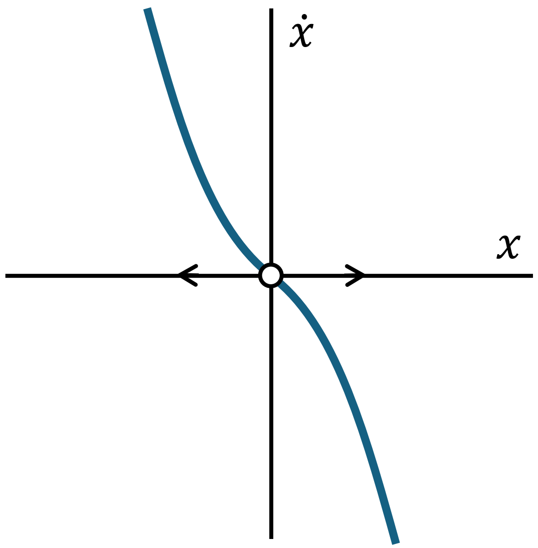
- Dynamical System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda)$
- $\lambda = \lambda_0$ is a bifurcation point if the topological structure of the dynamical system changes as λ passes through λ_0

- Example (Subcritical Pitchfork):

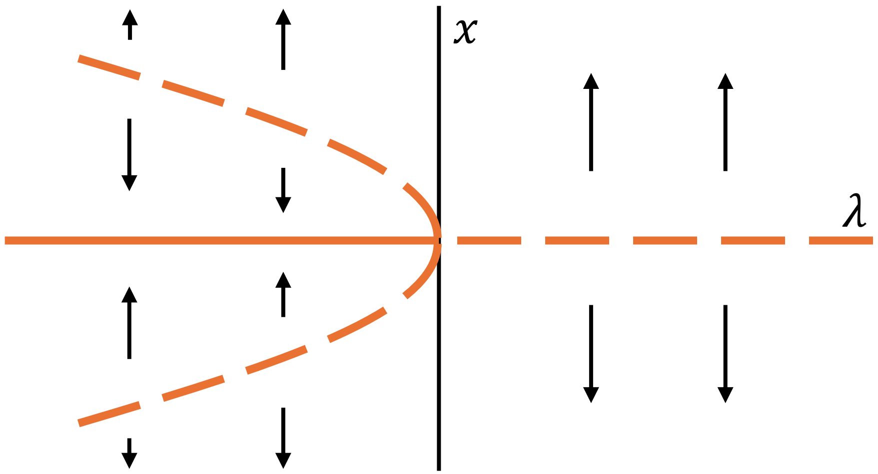
$$\dot{x} = \lambda x + x^3$$



$\lambda < 0$

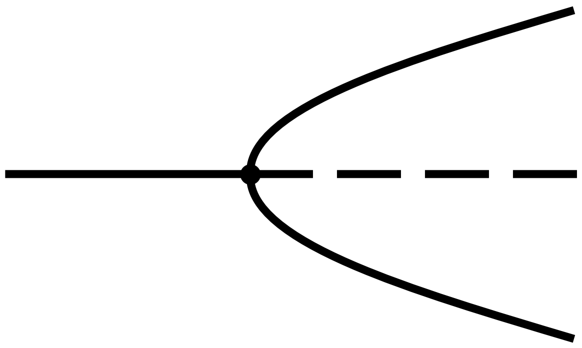


$\lambda > 0$

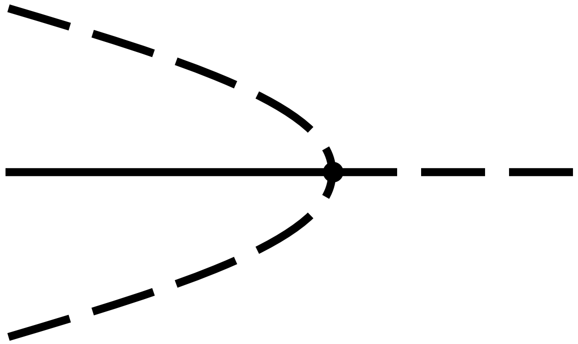


Bifurcation Overview

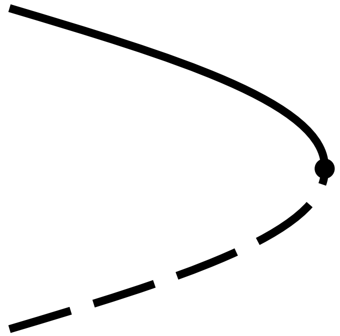
- Stationary Bifurcations



Supercritical Pitchfork

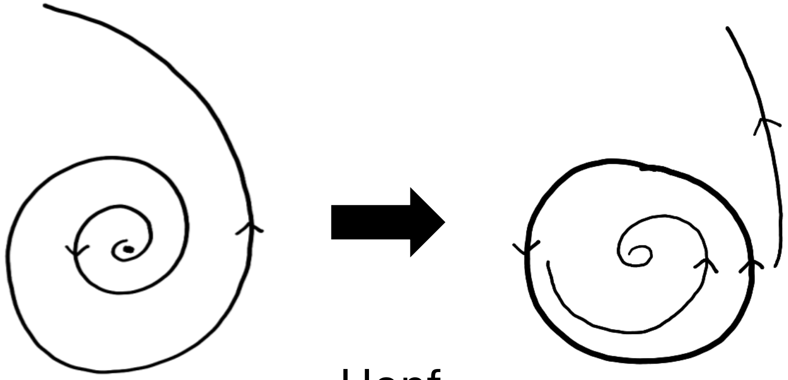


Subcritical Pitchfork



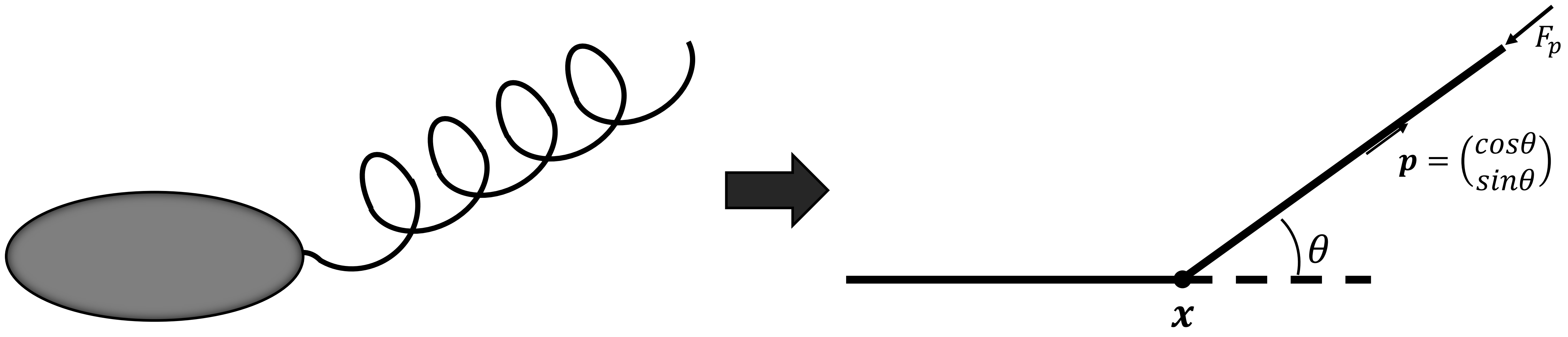
Saddle-Node

- Periodic Bifurcations



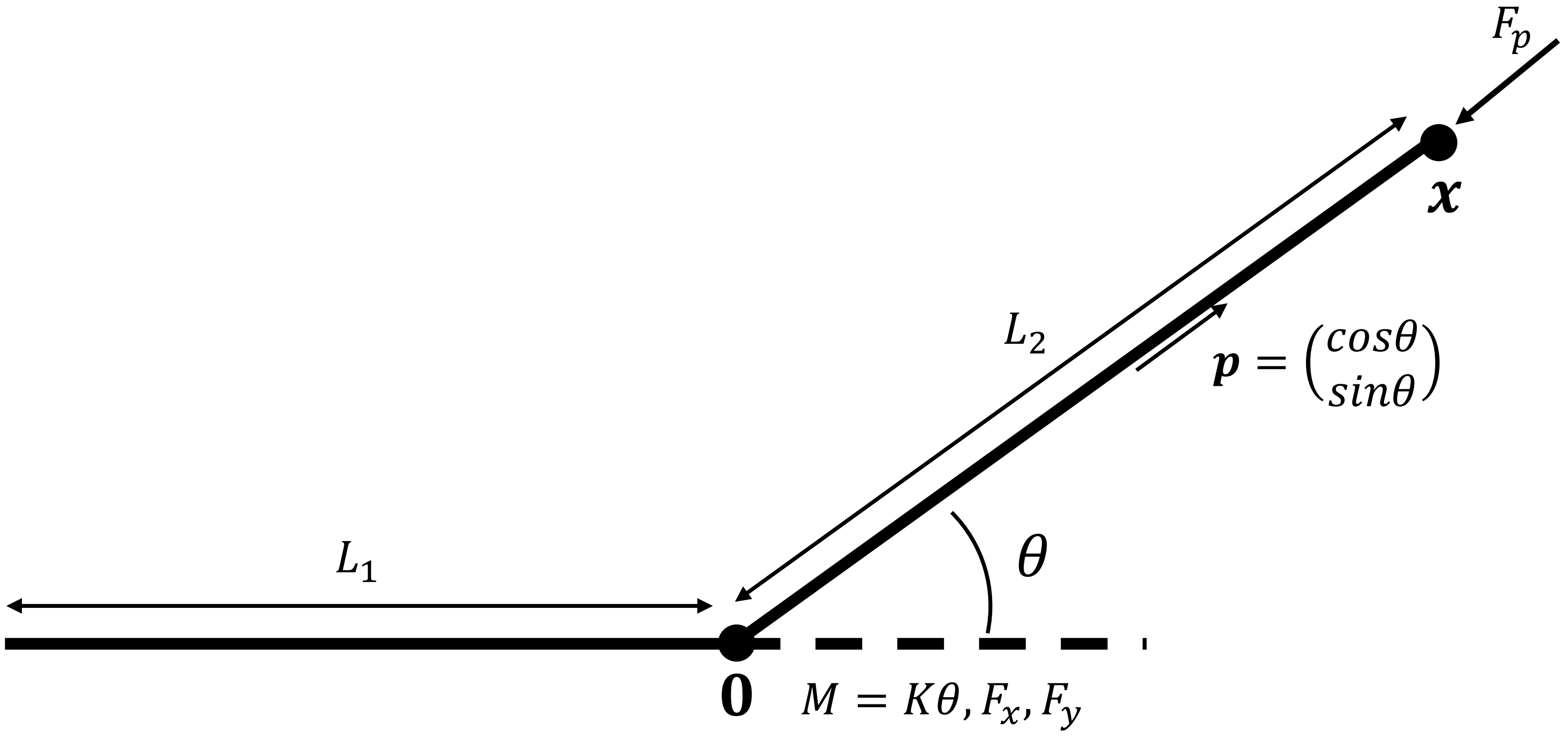
Hopf

The Model



- Head/Flagellum – slender rod
- Propulsive force parallel to flagellum
- Hook joining the two rods:
  1. Linear Spring
  2. Kirchhoff Rod
- Resistive Force Theory and Force Balancing for equations of motion

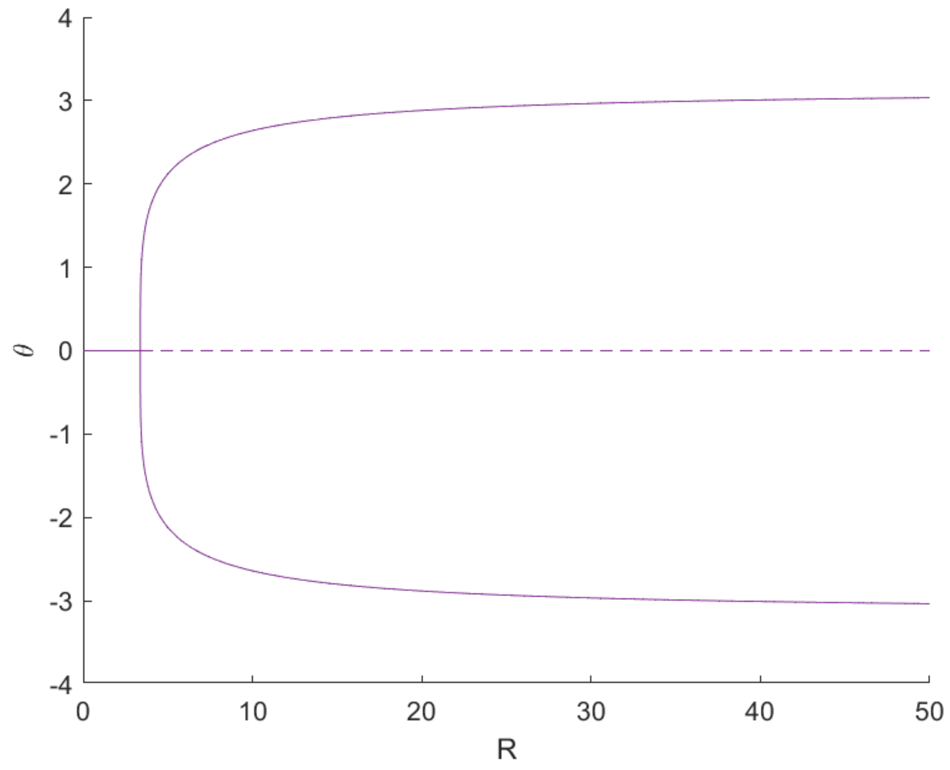
Linear Spring Model - Description



- Introduce dimensionless parameters: $L = \frac{L_2}{L_1}$, $R = \frac{F_p L_1}{K}$
- Obtain dynamical system: $\dot{\theta} = f(\theta, L, R)$

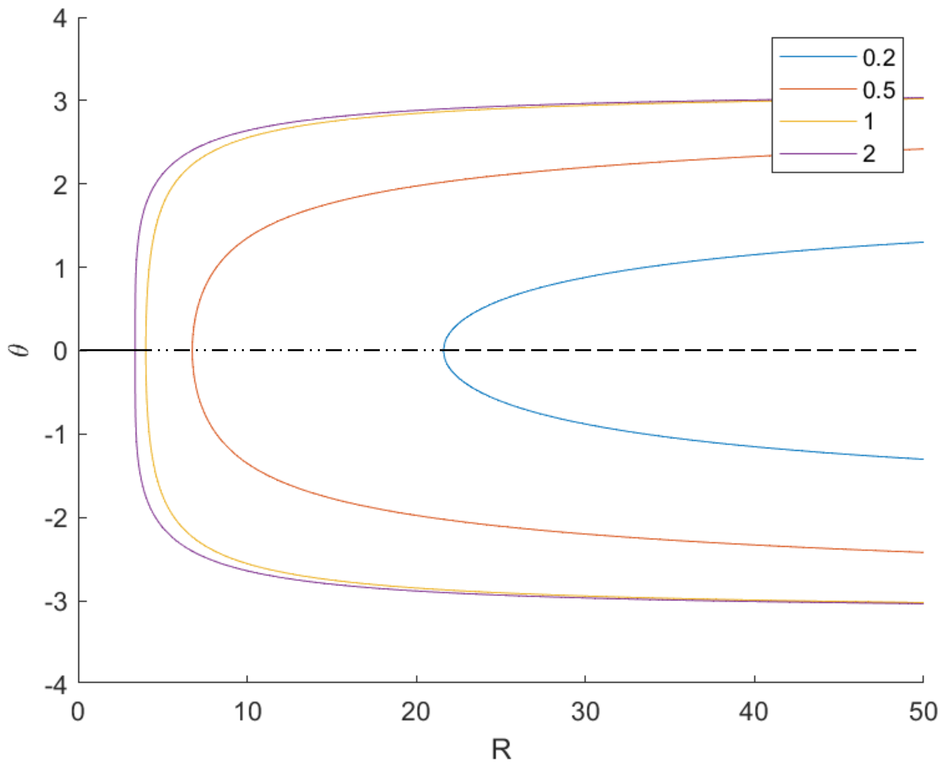
Linear Spring Model - Bifurcations

- $\theta = 0$ always fixed point
- Linearisation: $\dot{\theta} = -\frac{3\left((L+1)^3 - 2L^2 R\right)}{4L^3 R}\theta$
- Bifurcation point at $R_c = \frac{(L+1)^3}{2L^2}$
  - Stable for $R < R_c$
  - Unstable for $R > R_c$
- Cubic order terms show supercritical pitchfork bifurcation

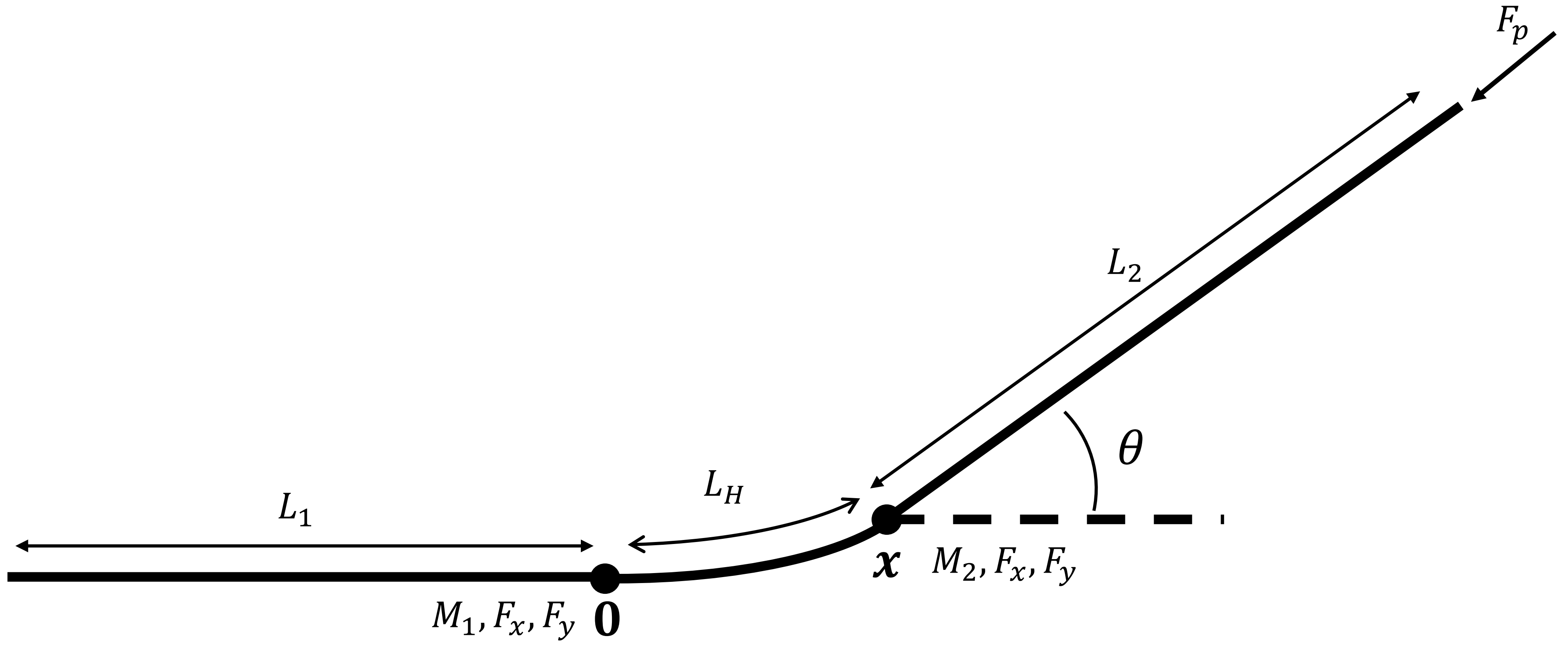


Linear Spring Model - Bifurcations

- $\theta = 0$ always fixed point
- Linearisation: $\dot{\theta} = -\frac{3\left((L+1)^3 - 2L^2 R\right)}{4L^3 R}\theta$
- Bifurcation point at $R_c = \frac{(L+1)^3}{2L^2}$
  - Stable for $R < R_c$
  - Unstable for $R > R_c$
- Cubic order terms show supercritical pitchfork bifurcation



Kirchhoff Rod Model - Description



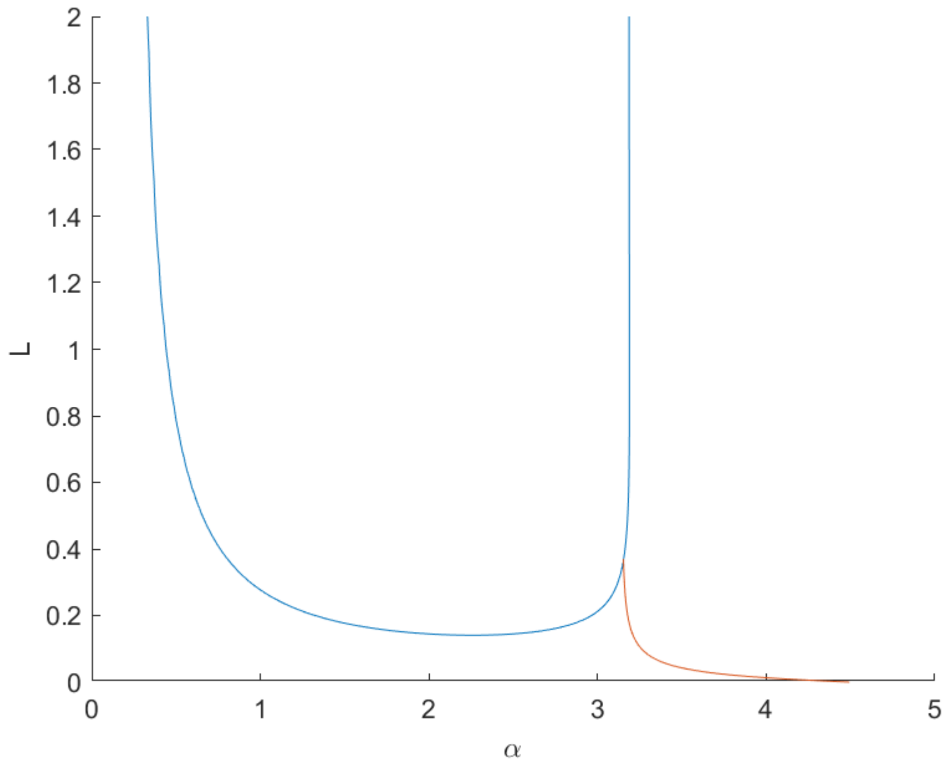
- Kirchhoff Rod Equations for Forces/Moments in terms of position
- Introduce dimensionless parameters: $L = \frac{L_2}{L_1}$, $L_h = \frac{L_H}{L_1}$, $R = \frac{F_p L_1 L_H}{EI}$
- No analytic solution of form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, L, L_h, R)$

Kirchhoff Rod Model - Linearisation

- Linearise equations about straight swimming
- Obtain relationship

$$\begin{pmatrix} \dot{y} \\ \dot{\theta} \end{pmatrix} = A_{(\alpha, L_h, L)} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

- Stationary bifurcation when $\det(A) = 0$
- Hopf bifurcation when $\mathrm{trace}(A) = 0$, $\det(A) > 0$



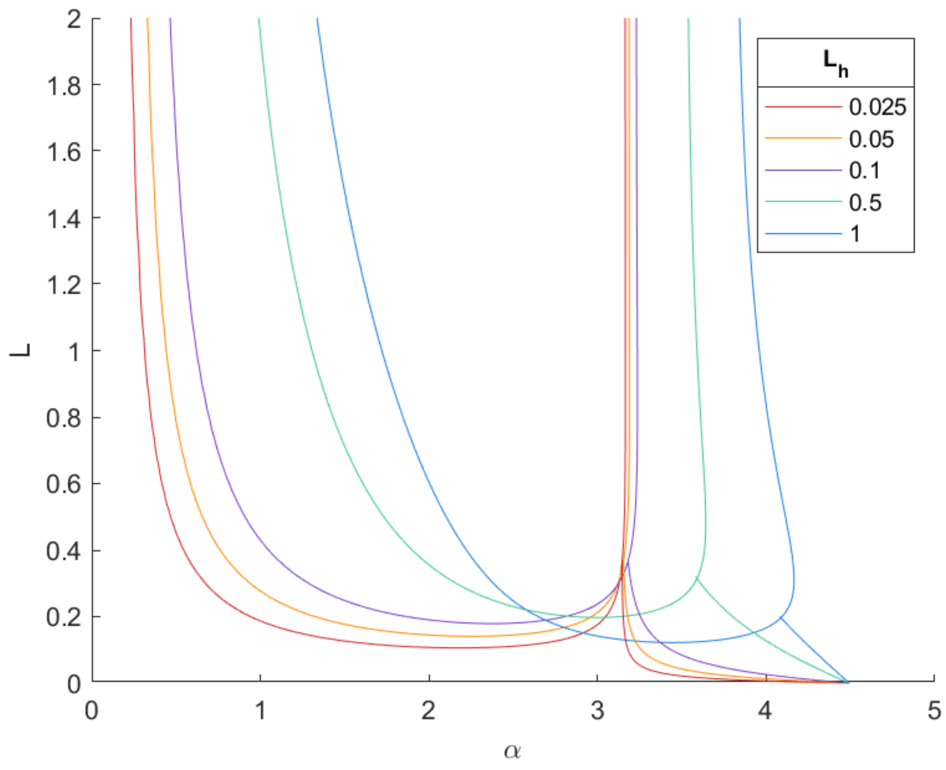
$$\alpha^2 = \frac{RL_h}{L+1}$$

Kirchhoff Rod Model - Linearisation

- Linearise equations about straight swimming
- Obtain relationship

$$\begin{pmatrix} \dot{y} \\ \dot{\theta} \end{pmatrix} = A_{(\alpha, L_h, L)} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

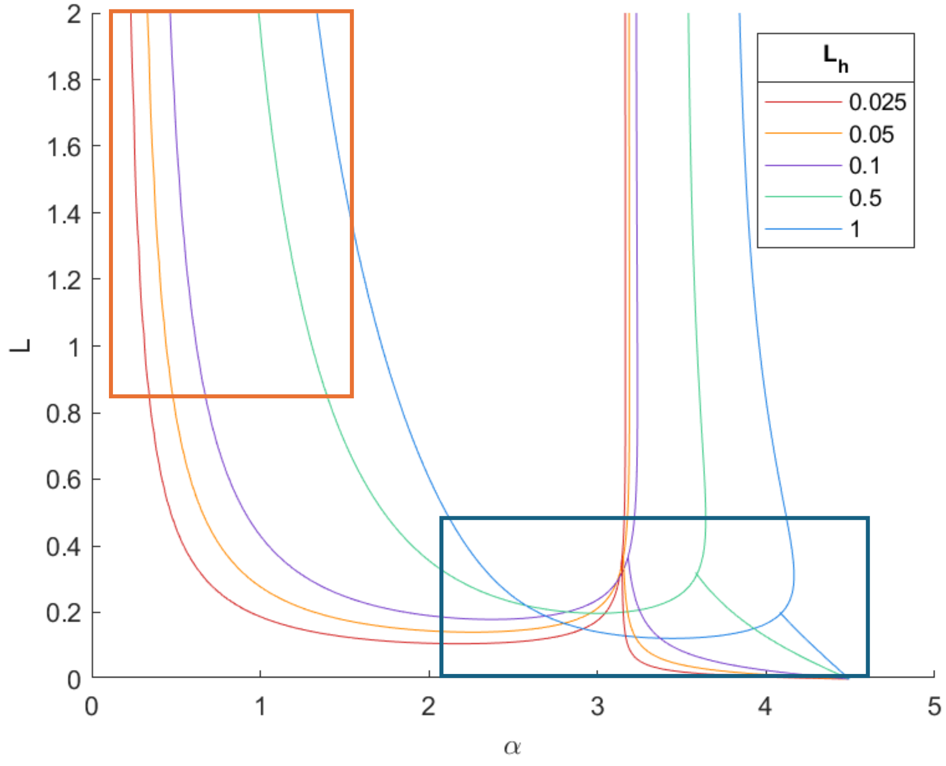
- Stationary bifurcation when $\det(A) = 0$
- Hopf bifurcation when $\mathrm{trace}(A) = 0$, $\det(\mathrm{A}) > 0$



$$\alpha^2 = \frac{RL_h}{L+1}$$

Kirchhoff Rod Model - Linearisation

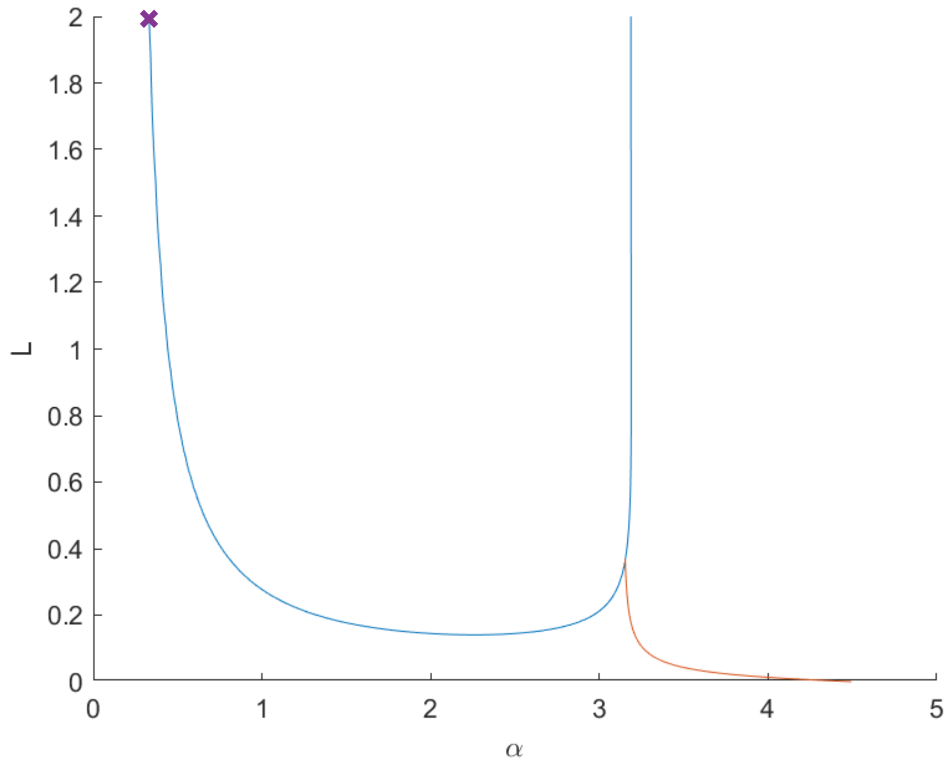
- Linearise equations about straight swimming
- Obtain relationship
$$\begin{pmatrix}\dot{y}\\ \dot{\theta}\end{pmatrix} = A_{(\alpha,L_h,L)}\begin{pmatrix}y\\ \theta\end{pmatrix}$$
- Stationary bifurcation when $\det(A) = 0$
- Hopf bifurcation when $\mathrm{trace}(A) = 0$, $\det(\mathrm{A}) > 0$

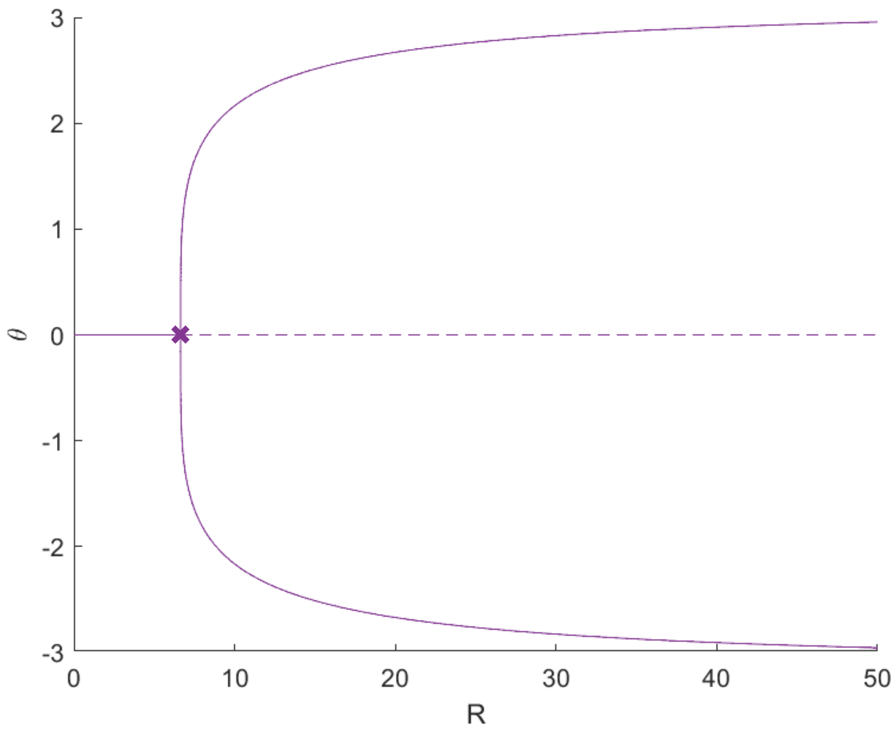


$$\alpha^2 = \frac{RL_h}{L+1}$$

Kirchhoff Rod Model - Bifurcations

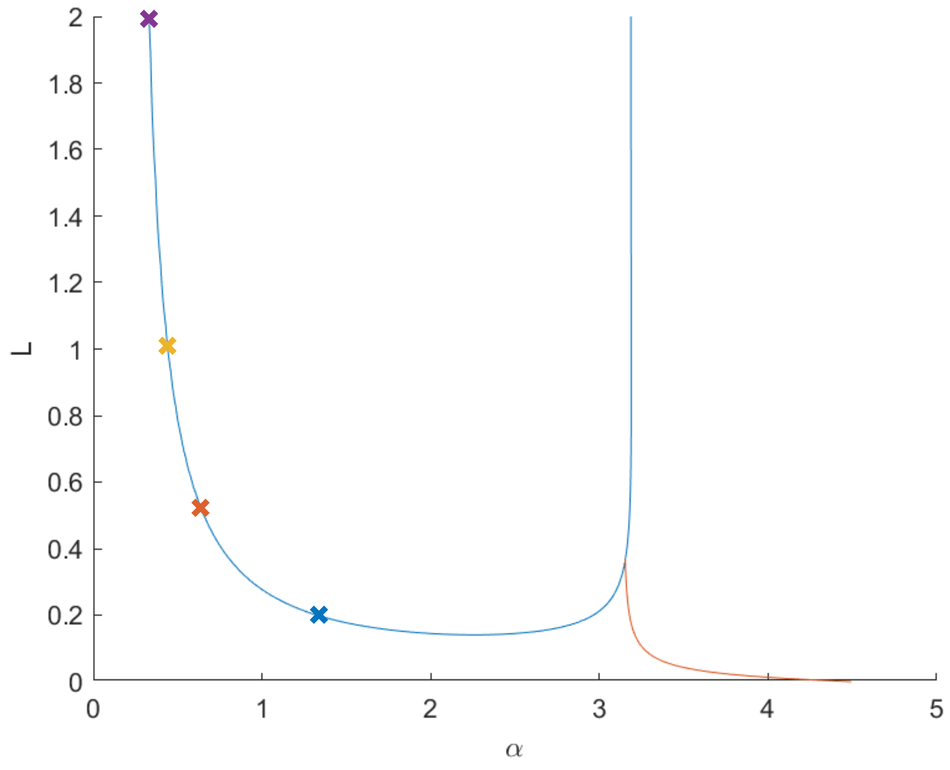
- First Stationary Bifurcation - Supercritical Pitchfork

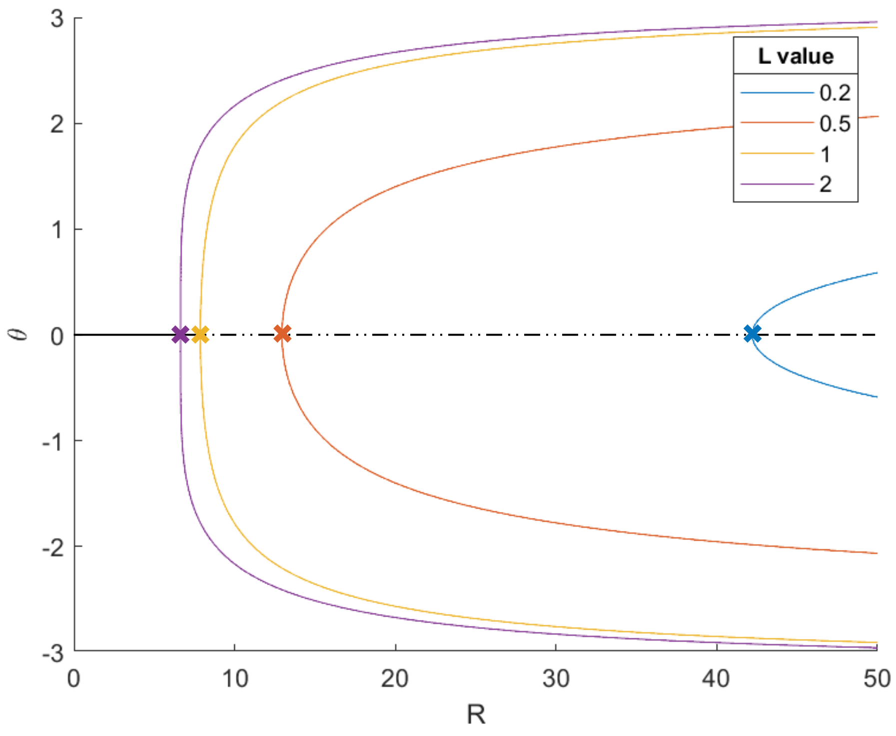




Kirchhoff Rod Model - Bifurcations

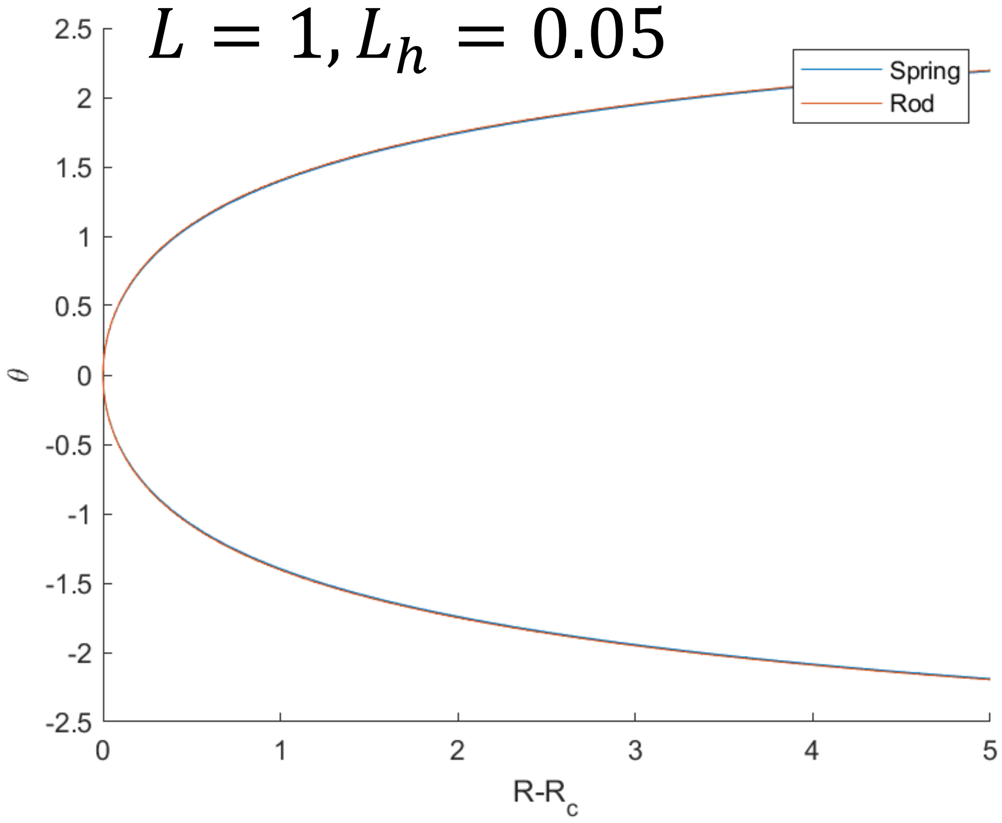
- First Stationary Bifurcation - Supercritical Pitchfork

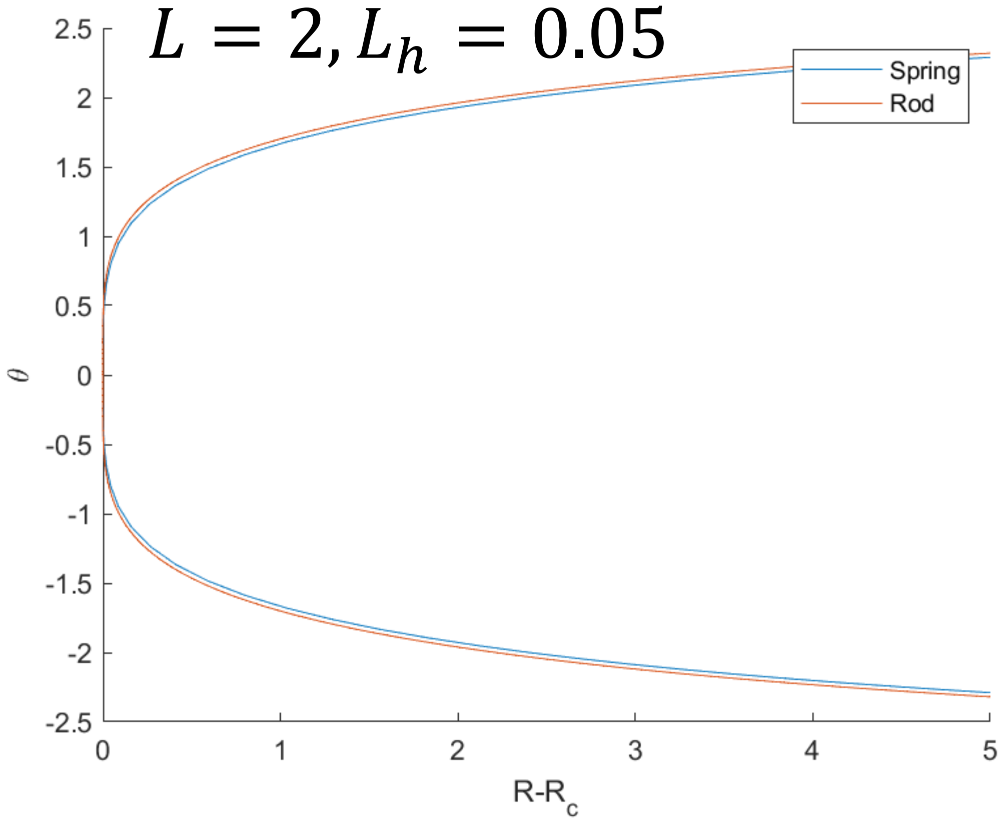




Bifurcation Comparison

- $R_r = 2R_s$
- R_c differs slightly between models
- Pitchfork takes similar shape in both models

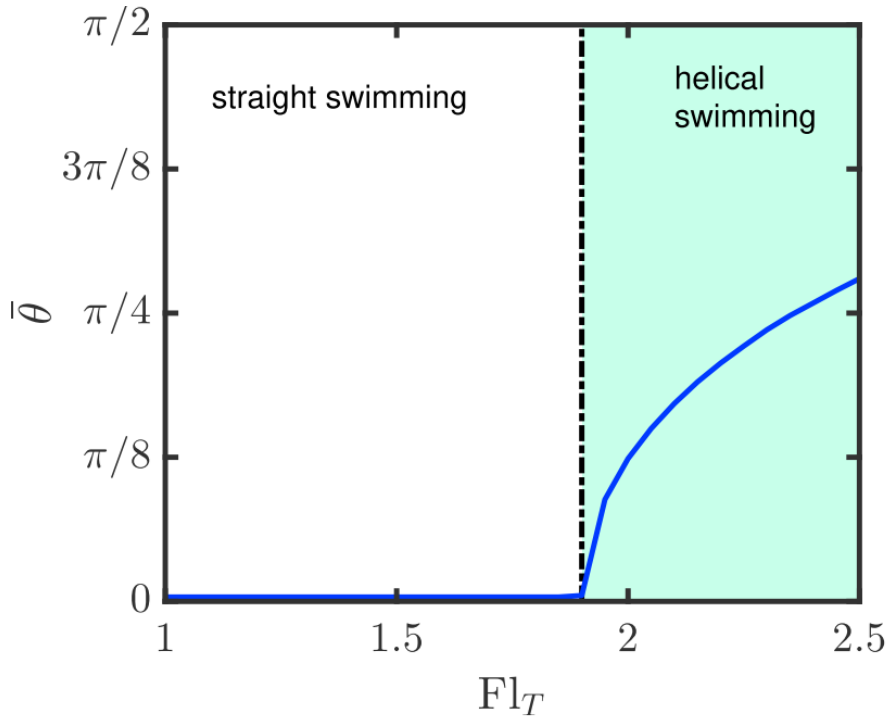




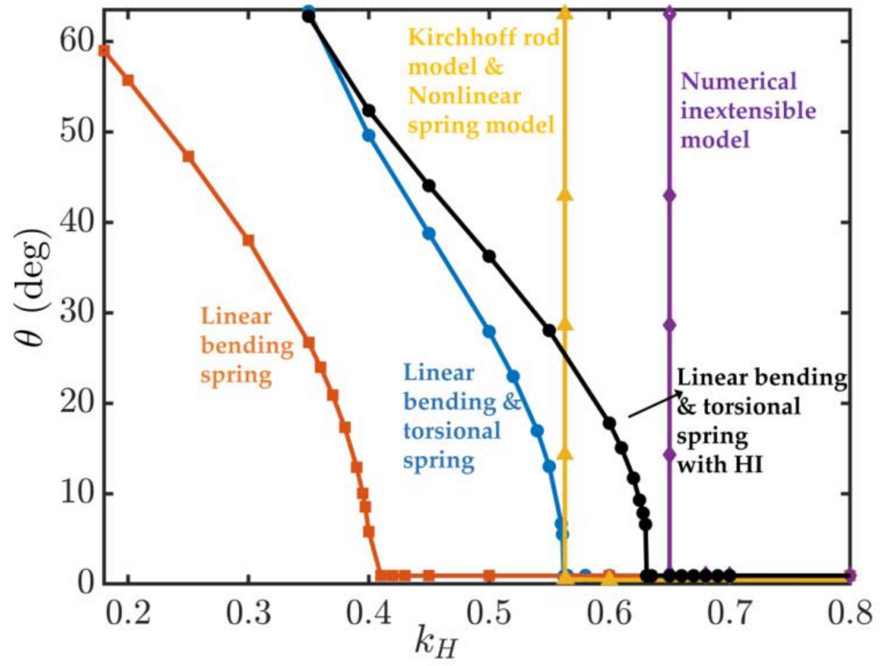
Comparison with other work

- All papers agree on Supercritical Bifurcation for spring model
- Not all papers agree on Supercritical Bifurcation for Kirchhoff Rod Model

Nguyen and Graham, 2017



Jabbarzadeh and Fu, 2020



Thank you