Monads and distributive laws

Maia Woolf

University of Cambridge

2024 August 21

Overview

- What is a monad?

- What is a distributive law?

Monads

Let **Sets** be the category of all sets.

A *monad* on **Sets** is a functor (think function) $T : \textbf{Sets} \to \textbf{Sets}$ with some extra structure.

Monads

Example (Free abelian group monad)

$A:$ **Sets** $\rightarrow$ **Sets**

$\{a, b, c\} \mapsto \{0, \ a, \ b, \ c, \ 2a, \ a - 7b, \ -a + b + 4c, \ \dots\}$

An abelian group is just a set X and a function $A(X) \rightarrow X$ satisfying some properties.

Monads

Example (Free monoid monad)

$$M : \quad \textbf{Sets} \rightarrow \textbf{Sets}$$
$$\{a, b, c\} \mapsto \{1, a, b, c, aa, abba, bbbac, \dots\}$$

A monoid is just a set X and a function $M(X) \rightarrow X$ satisfying some properties.

Monads

What if we apply a monad twice?

For monoids:

$$MM(\{a, b\}) = \{(bbb)(a),\ (ab)(ba),\ (abb)(a),\ (ba)(1)(a),\ \dots\}$$

$$\downarrow$$

$$M(\{a, b\}) = \{\ bbba\ ,\quad abba\ =\ abba\ ,\quad baa\quad ,\ \dots\}$$

Monads

What if we apply a monad twice?

For monoids:

$$MM(\{a, b\}) = \{(bbb)(a), \ (ab)(ba), \ (abb)(a), \ (ba)(1)(a), \ \dots\}$$

$$\downarrow$$

$$M(\{a, b\}) = \{ \quad bbba \quad , \quad abba \ = \ abba \quad , \quad baa \quad , \dots\}$$

It is part of the structure of all monads T that we can map
$TT(X) \to T(X)$ to 'simplify terms of terms'.

Monads

Can we compose monads?

E.g. can we make $AM : \mathbf{Sets} \to \mathbf{Sets}$ into a monad?

Well, we'd need a way to map

$$AMAM(X) \to AM(X).$$

Distributive laws

To get a function

$$AMAM(X) \to AM(X),$$

we use something called a *distributive law*.

A distributive law here is a way to transform

$$\lambda : MA \to AM.$$

Distributive laws

To get a function

$$AMAM(X) \rightarrow AM(X),$$

we use something called a *distributive law*.

A distributive law here is a way to transform

$$\lambda : MA \rightarrow AM.$$

Then we can do

$$AMAM(X) \xrightarrow{\lambda} AAMM(X) \xrightarrow{\text{simplify}} AM(X).$$

Distributive laws

Example

$$\lambda : \qquad MA \to AM$$
$$(a + b)(c + d) \mapsto ac + ad + bc + bd$$

Distributive laws

Example

$$\lambda : \qquad MA \to AM$$
$$(a + b)(c + d) \mapsto ac + ad + bc + bd$$

This distributive law lets us simplify elements of $AMAM(X)$ to elements of $AM(X)$:

$$(aa + 1)(b + 1) + (b + 1)(1 + a)$$
$$\xmapsto{\lambda} ((aa)(b) + (aa)(1) + (1)(b) + (1)(1))$$
$$+ ((b)(1) + (b)(a) + (1)(1) + (1)(a))$$
$$\xmapsto{\text{simplify}} aab + aa + ba + a + 2b + 2.$$

Distributive laws

Example

$$\lambda : \qquad MA \to AM$$
$$(a + b)(c + d) \mapsto ac + ad + bc + bd$$

This distributive law lets us simplify elements of $AMAM(X)$ to elements of $AM(X)$:

$$(aa + 1)(b + 1) + (b + 1)(1 + a)$$
$$\overset{\lambda}{\longmapsto} ((aa)(b) + (aa)(1) + (1)(b) + (1)(1))$$
$$+ ((b)(1) + (b)(a) + (1)(1) + (1)(a))$$
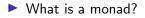$$\overset{\text{simplify}}{\longmapsto} aab + aa + ba + a + 2b + 2.$$

This makes AM into the free ring monad!

Distributive laws

How many distributive laws $MA \rightarrow AM$ are there?

It turns out that there is only one, the λ from before giving multiplication distributing over addition (Zwart-Marsden, 2018).

My work has included looking for features of distributive laws that limit the forms they can take.

Overview

- What is a monad?

- What is a distributive law?

Thank you!

My thanks to Martin Hyland, for his sage advice, support, and for being an all-round wonderful supervisor.

My thanks to all of the other summer interns, too numerous to name, that I have spent time with, who have made this a very enjoyable summer; and to Fanny Seizilles for being a fab mentor.

My thanks to Sally Baume and the other organisers who made this such a supportive and enriching programme.

Philippa Fawcett (1868-1948)

Hail the triumph of the corset
Hail the fair Philippa Fawcett
Victress in the fray
    Crown her queen of Hydrostatics
    And the other Mathematics
Wreathe her brow with bay.

– Anonymous