Asymptotic Decomposition of a Scalar Field in de Sitter Space

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Motivation

Einstein's equations of general relativity:

$$\underbrace{R_{ab} - \frac{1}{2}R g_{ab} + \overset{\text{Cosmological constant}}{\lambda}\overset{\text{Spacetime metric}}{g_{ab}}}_{\text{Curvature of spacetime}} = \underbrace{8\pi G\, T_{ab}}_{\text{Energy-momentum of spacetime}}.$$

De Sitter space = Maximally symmetric solution of Einstein's equations with positive cosmological constant.

Goal: To investigate the existence of a conjectured asymptotic expansion for the charged scalar field on de Sitter space:

$$\phi \sim \varphi_1\, \mathrm{e}^{-Ht} + \varphi_2\, \mathrm{e}^{-2Ht} + \varphi_3\, \mathrm{e}^{-3Ht} + \dots$$

De Sitter Space

De Sitter space dS_4 may be defined as the hyperboloid

$$|x|^2 - x_0^2 = \frac{1}{H^2}$$

in $(4+1)$-dimensional Minkowski space

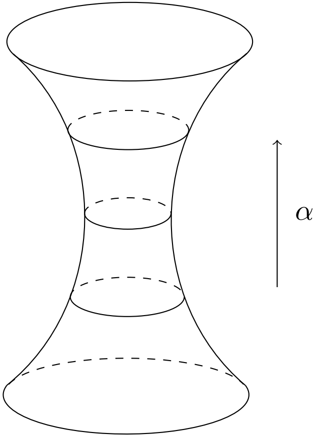
$$\eta_5 = \mathrm{d}x_0^2 - \mathrm{d}|x|^2 - |x|^2 g_{\mathbb{S}^3}.$$

Defining

$$x_0 = \frac{1}{H}\sinh(H\alpha), \qquad |x| = \frac{1}{H}\cosh(H\alpha),$$

the metric η_5 descends to the metric g on dS_4,

$$\boxed{g = \mathrm{d}\alpha^2 - \frac{1}{H^2}\cosh^2(H\alpha) g_{\mathbb{S}^3}.}$$



Conformal Compactification

To study the asymptotic structure of a spacetime $(\mathscr{M}, g)$ at infinity, we make the *conformal transformation*

$$g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}$$

Conformal factor, $\to 0$ asymptotically

This brings infinity to a finite region.

Attach to $\mathscr{M}$ a boundary $\mathscr{I} := \{\Omega = 0\}$ and get a new spacetime

$$\hat{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$$

Asymptotic considerations in physical spacetime $\mathscr{M}$

$$\updownarrow$$

Local differential geometry near $\mathscr{I}$ in the rescaled spacetime $\hat{\mathscr{M}}$.

Conformal Compactification of de Sitter Space

$$g = \mathrm{d}\alpha^2 - \frac{1}{H^2}\cosh^2(H\alpha) g_{\mathbb{S}^3}$$

Make the coordinate transformation

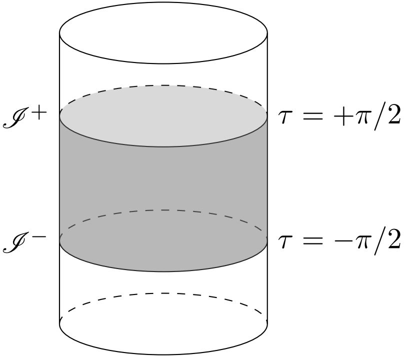
$$\tan\left(\frac{\tau}{2}\right) = \tanh\left(\frac{H\alpha}{2}\right)$$

so that the metric becomes

$$g = \underbrace{\frac{1}{H^2\cos^2\tau}}_{\Omega^{-2}} \underbrace{(\mathrm{d}\tau^2 - g_{\mathbb{S}^3})}_{\hat{g}}$$

where $\tau \in (-\pi/2, \pi/2)$.

↑
Metric on the
Einstein cylinder
$\mathbb{R} \times \mathbb{S}^3$



Conformal Compactification of de Sitter Space

$$g = \Omega^{-2}(\mathrm{d}\tau^2 - g_{\mathbb{S}^3}), \quad \Omega = H\cos\tau$$

We can attach to $(-\pi/2, \pi/2) \times \mathbb{S}^3$ the boundary

$$\mathscr{I} := \{\Omega = 0\} = \{\tau = \pm\pi/2\}$$

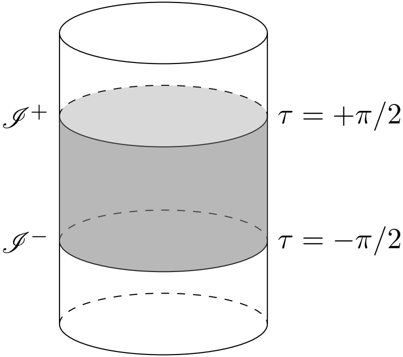
and identify compactified de Sitter space $\widehat{\mathrm{dS}}_4$ with $[-\pi/2, \pi/2] \times \mathbb{S}^3$.

The boundary is the union of the spacelike hypersurfaces

$$\mathscr{I}^+ = \left\{\tau = +\frac{\pi}{2}\right\}, \qquad \mathscr{I}^- = \left\{\tau = -\frac{\pi}{2}\right\}.$$

Future null infinity

Past null infinity



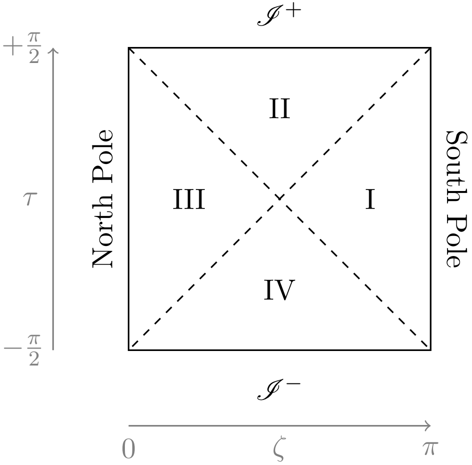
Penrose Diagram for de Sitter Space

$$\boxed{g = \Omega^{-2}(\mathrm{d}\tau^2 - g_{\mathbb{S}^3}), \quad \Omega = H\cos\tau}$$

If we write the three-sphere metric as

$$g_{\mathbb{S}^3} = \mathrm{d}\zeta^2 + (\sin^2\zeta)g_{\mathbb{S}^2}$$

for $\zeta \in [0,\pi]$ and quotient out the SO(3) symmetry group of $g_{\mathbb{S}^2}$, we obtain the Penrose diagram for dS_4.



Static Coordinates on de Sitter Space

Static coordinates on dS_4 may be constructed by defining

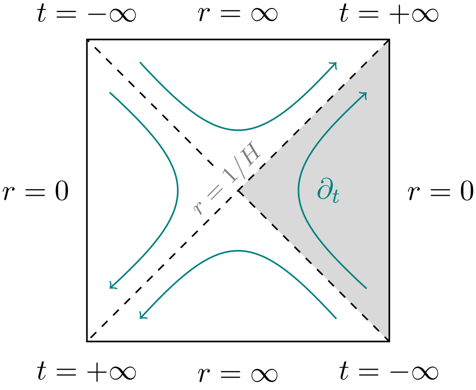
$$r = \frac{\sin\zeta}{H\cos\tau}, \qquad \tanh(Ht) = \frac{\sin\tau}{\cos\zeta}$$

for $\tau \in (-\pi/2, \pi/2)$ and $\zeta \in (0, \pi)$.

Then

$$g = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2 g_{\mathbb{S}^2},$$

where $F(r) = 1 - H^2r^2$.



The Conformal Wave Equation

For a generic spacetime $(\mathscr{M}, g)$, the conformal wave equation is

Scalar field

$$\Box\phi + \frac{1}{6}R\phi = 0.$$

$\nabla_a\nabla^a = g^{ab}\nabla_a\nabla_b$

Scalar curvature of spacetime

Consider the conformal transformation $\hat{g}_{ab} = \Omega^2 g_{ab}$, and choose

$$\hat{\phi} := \Omega^{-1}\phi.$$

Then the wave equation is *conformally invariant*:

$$\Box\phi + \frac{1}{6}R\phi = 0 \quad \Longleftrightarrow \quad \hat{\Box}\hat{\phi} + \frac{1}{6}\hat{R}\hat{\phi} = 0.$$

The Conformal Wave Equation on de Sitter Space

For de Sitter space we have $R = 12H^2$, so that the wave equation on dS_4 is

$$\Box\phi + 2H^2\phi = 0.$$

Under the rescaling

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \qquad \hat{\phi} = \Omega^{-1}\phi, \qquad \text{with } \ \Omega = H\cos\tau,$$

this becomes the conformal wave equation on the Einstein cylinder,

$$\hat{\Box}\hat{\phi} + \hat{\phi} = 0.$$

The Conformal Method

Estimates for $\hat{\phi}$ on compactified spacetime $\widehat{\mathrm{dS}}_4$

$$\downarrow$$

Estimates for ϕ on physical spacetime dS_4

Decay Estimate

Estimates for $\hat{\phi}$ on Einstein cylinder
$\rightarrow$ Estimates for ϕ on physical spacetime dS_4.

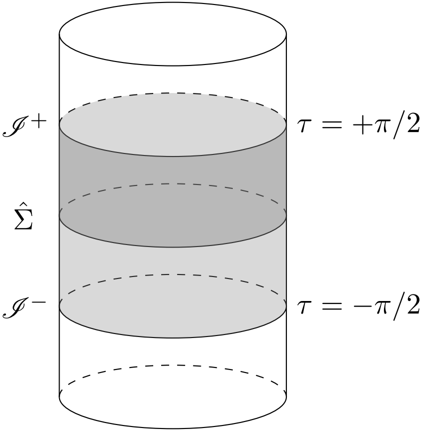
For sufficiently regular initial data $(\hat{\phi}, \partial_\tau \hat{\phi})|_{\hat{\Sigma}}$, one can show that

$$|\hat{\phi}| \leq C \quad \text{as} \quad \tau \to \pi/2.$$

Then since $\phi = \Omega\, \hat{\phi}$,

$$|\phi| \underset{\uparrow}{\lesssim} \Omega \quad \text{as} \quad t \to +\infty.$$

Inequality up to a constant



Decay Estimate in Static Coordinates

In the static coordinates,

$$\Omega = \frac{H}{\cosh(Ht)} \frac{1}{\sqrt{1 - H^2 r^2 \tanh^2(Ht)}} \sim \frac{H \mathrm{e}^{-Ht}}{\sqrt{1 - H^2 r^2}} \text{ as } t \to +\infty,$$

so that keeping r fixed, we have

$$|\phi| \lesssim \Omega \lesssim_r \mathrm{e}^{-Ht} \text{ as } t \to +\infty.$$

Asymptotic Decomposition of a Scalar Field

We now know that

$$\phi \sim \varphi_1 \mathrm{e}^{-Ht} + \mathcal{O}(\mathrm{e}^{-2Ht}) \quad \text{as} \;\; t \to +\infty.$$

How can we find the coefficient φ_1?

Direct substitution into the conformal wave equation:

$$\Box = F(r)^{-1}\partial_t^2 - \frac{1}{r^2}\partial_r(r^2 F(r)\partial_r) - \frac{1}{r^2}\nabla^2_{\mathbb{S}^2}$$

Reminder:

$$F(r) = 1 - H^2 r^2$$

$$\begin{aligned} 0 &= \hat{\Box}\hat{\phi} + 2H^2\hat{\phi} \\ &\sim \mathrm{e}^{-Ht}\left[\left(F(r)^{-1} + 2\right) H^2 \varphi_1 - \frac{1}{r^2}\partial_r\left(r^2 F(r)\partial_r \varphi_1\right) - \frac{1}{r^2}\nabla^2_{\mathfrak{s}_2}\varphi_1\right] \\ &\text{as } t \to +\infty. \end{aligned}$$

Asymptotic Decomposition: First Coefficient

Seperating variables by writing $\varphi_1 = F^{-\frac{1}{2}} R_1(r)\Theta_1(\omega^{(2)})$, we obtain

$$\nabla^2_{\mathfrak{s}_2}\Theta_n + \lambda\Theta_n = 0,$$

$$\frac{\mathrm{d}^2 R_1}{\mathrm{d}z^2} + \frac{\mathrm{d}R_1}{\mathrm{d}z}\left(\frac{2}{z}\right) + R_1\left(-\frac{\lambda}{z^2} + \frac{\frac{1}{2} - \lambda}{z+1} + \frac{\frac{1}{2}\lambda}{z-1}\right) = 0.$$

where $z := Hr$. The spherical component is solved by the spherical harmonics $Y_{l,m}$.

$$\sigma^2 + \sigma + l(l+1) = 0.$$

$$R_{1,n,m,l} = z^l \sum_{k=0}^{\infty} a_k z^k,$$

$$R_{2,n,m,l} = R_{1,n,m,l}\log z + \sum_{k=0}^{\infty} b_k z^{k-l-1}.$$

Asymptotic Decomposition

Hence, we have

$$\begin{aligned}\phi &\sim \sum_{l=0}^{1}\sum_{m=-l}^{l}\alpha_{1,m,l}R_{1,n,m,l}Y_{l,m}F(r)^{-1/2}\mathrm{e}^{-Ht}+\mathcal{O}(e^{-2Ht})\\ &= a_0F(r)^{-1/2}\mathrm{e}^{-Ht}+\mathcal{O}(e^{-2Ht})\quad \text{as } t\to+\infty.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\phi &\sim \sum_{n=1}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l}\alpha_{n,m,l}R_{1,n,m,l}Y_{l,m}F(r)^{-n/2}\mathrm{e}^{-nHt}\\ &= \sum_{n=1}^{\infty}P_n(r)F(r)^{-n/2}\mathrm{e}^{-nHt}\quad \text{as } t\to+\infty.\end{aligned}$$

where P_n is a polynomial in Hr of degree $n-1$.

Asymptotic Decomposition of a Scalar Field

We now know that

$$\phi \sim \varphi_1 \mathrm{e}^{-Ht} + \mathcal{O}(\mathrm{e}^{-2Ht}) \quad \text{as} \ \ t \to +\infty.$$

How can we find the coefficient φ_1?

Relate derivatives on $\widehat{\mathrm{dS}}_4$ to derivatives on dS_4:

$$\begin{aligned}\Omega\partial_\zeta\hat{\phi} &= \frac{\partial t}{\partial\zeta}\partial_t\phi + \frac{\partial r}{\partial\zeta}\partial_r\phi \\ &= rF(r)^{-1/2}\sinh(Ht)\partial_t\phi + H^{-1}F(r)^{1/2}\cosh(Ht)\partial_r\phi\end{aligned}$$

$$\begin{aligned}\Omega\partial_\tau\hat{\phi} &= \frac{\partial t}{\partial\tau}\partial_t\phi + \frac{\partial r}{\partial\tau}\partial_r\phi - \Omega^{-1}(\partial_\tau\Omega)\phi \\ &= H^{-1}F(r)^{-1/2}\cosh(Ht)\partial_t\phi + rF(r)^{1/2}\sinh(Ht)\partial_r\phi + F(r)^{1/2}\sinh(Ht)\phi\end{aligned}$$

Reminder:

$$r = \frac{\sin\zeta}{H\cos\tau}$$

$$\tanh(Ht) = \frac{\sin\tau}{\cos\zeta}$$

Asymptotic Decomposition: First Coefficient

$$\Omega\partial_\zeta\hat{\phi} = rF(r)^{-1/2}\sinh(Ht)\partial_t\phi + H^{-1}F(r)^{1/2}\cosh(Ht)\partial_r\phi$$

$$\Omega\partial_\tau\hat{\phi} = H^{-1}F(r)^{-1/2}\cosh(Ht)\partial_t\phi + rF(r)^{1/2}\sinh(Ht)\partial_r\phi + F(r)^{1/2}\sinh(Ht)\phi$$

For sufficiently regular initial data, $\partial_\zeta\hat{\phi}$ and $\partial_\tau\hat{\phi}$ have continuous limits on $\mathscr{I}^+$, so

$$|\Omega\partial_\zeta\hat{\phi}|, |\Omega\partial_\tau\hat{\phi}| \lesssim \Omega \lesssim \mathrm{e}^{-Ht} \quad \text{as} \;\; t \to +\infty.$$

Considering the e^{-Ht} component of ϕ,

$$\varphi_1 := \mathrm{e}^{Ht}\phi,$$

and taking the limit as $t \to +\infty$,

$$0 \approx Hr\partial_t\varphi_1 - H^2 r\varphi_1 + F\partial_r\varphi_1,$$
$$0 \approx \partial_t\varphi_1 - H\varphi_1 + HrF\partial_r\varphi_1 + HF\varphi_1.$$

Equality at $t = +\infty$

Asymptotic Decomposition: First Coefficient

$$
\begin{aligned}
0 &\approx Hr\partial_t\varphi_1 - H^2 r\varphi_1 + F\partial_r\varphi_1, \\
0 &\approx \partial_t\varphi_1 - H\varphi_1 + HrF\partial_r\varphi_1 + HF\varphi_1
\end{aligned}
$$

Solving this algebraically, we find that $\partial_t\varphi_1 \approx 0$, and

$$H^2 r\varphi_1 \approx F(r)\partial_r\varphi_1.$$

Solving this ordinary differential equation in r, we obtain

$$\varphi_1(r) \approx \frac{1}{\sqrt{F(r)}}\varphi_1(0).$$

Conclusion

The conformal method can be used to study the asymptotic structures of spacetimes.

We investigated an asymptotic decomposition of a scalar field on de Sitter space:

$$\phi \sim \varphi_1 \, \mathrm{e}^{-Ht} + \varphi_2 \, \mathrm{e}^{-2Ht} + \varphi_3 \, \mathrm{e}^{-3Ht} + \dots \qquad \text{as } t \to \infty.$$

The coefficients are given by

$$\varphi_n(r) = \frac{a_0 + a_1 r + \dots + a_{n-1} r^{n-1}}{F(r)^{n/2}}.$$

Conclusion

The coefficients φ_n derived using the conformal method agree with

- Calculations using quasinormal modes on dS_4,
- Direct solution of the PDEs derived from the conformal wave equation.

The asymptotic expansion using the conformal method also holds for the non-linear Maxwell-scalar field system,

$$\nabla^b F_{ab} = \mathrm{Im}(\bar{\phi} D_a \phi),$$
$$D^a D_a \phi + \frac{1}{6} R \phi = 0.$$

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- Peter Hintz and YuQing Xie. 'Quasinormal modes and dual resonant states on de Sitter space'. *Physical Review D* 104.6 (Sept. 2021). arXiv: `2104.11810 [gr-qc]`.

Asymptotic Decomposition: Second Coefficient

For the second coefficient, compute

$$\Omega\partial_\zeta^2\hat{\phi}, \qquad \Omega\partial_\zeta\partial_\tau\hat{\phi}, \qquad \Omega\partial_\tau^2\hat{\phi},$$

and define

$$\varphi_2 := \mathrm{e}^{2Ht}(\phi - \varphi_1\mathrm{e}^{-Ht}).$$

We find that φ_2 is also independent of t, and obtain the ODE

$$F\partial_r^2\varphi_2 - 4H^2 r\partial_r\varphi_2 - 2H^2\varphi_2 \approx 0,$$

which has solution

$$\varphi_2(r) \approx \frac{\varphi_2(0) + r\varphi_2'(0)}{F(r)}.$$

Asymptotic Decomposition: Third Coefficient

Similarly, for the third coefficient, we compute the third derivatives

$$\Omega\partial_\zeta^3\hat{\phi}, \qquad \Omega\partial_\zeta^2\partial_\tau\hat{\phi}, \qquad \Omega\partial_\zeta\partial_\tau^2\hat{\phi} \qquad \Omega\partial_\tau^3\hat{\phi},$$

and find that

$$\varphi_3(r) \approx \frac{\varphi_3(0) + r\varphi_3'(0) + r^2\varphi_3''(0)}{F(r)^{3/2}}.$$

We thus have the asymptotic decomposition

$$\begin{aligned}\phi &\sim \varphi_1\,\mathrm{e}^{-Ht} + \varphi_2\,\mathrm{e}^{-2Ht} + \varphi_3\,\mathrm{e}^{-3Ht} + \dots \\ &\sim \frac{\varphi_1(0)}{F(r)^{1/2}}\mathrm{e}^{-Ht} + \frac{\varphi_2(0) + r\varphi_2'(0)}{F(r)}\mathrm{e}^{-2Ht} + \frac{\varphi_3(0) + r\varphi_3'(0) + r^2\varphi_3''(0)}{F(r)^{3/2}}\mathrm{e}^{-3Ht} + \dots\end{aligned}$$