# Learned Regularisation using Bregman-Moreau Envelope:

## two ways of approximating proximal operators

supervised by Zakhar Shumaylov and Ander Biguri

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# Special thanks:

CAMBRIDGE IMAGE ANALYSIS
DAMTP - UNIVERSITY OF CAMBRIDGE


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# Motivation – Computer Tomography




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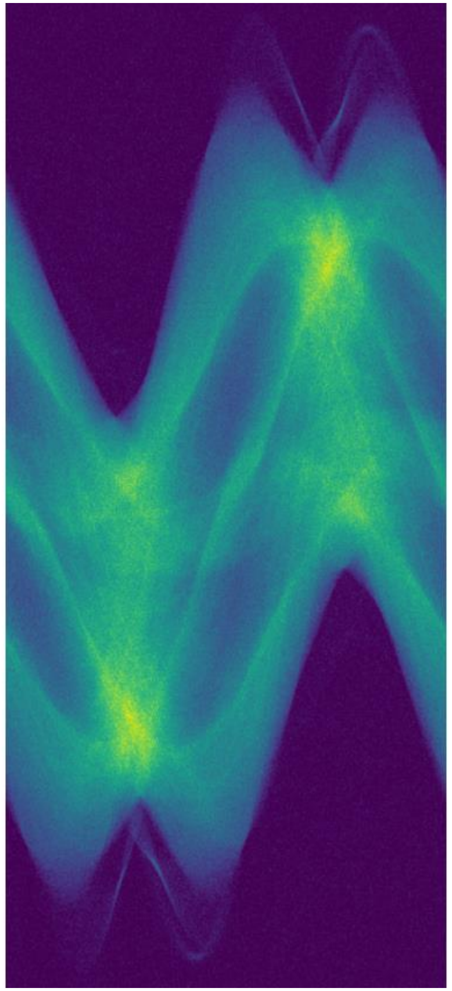
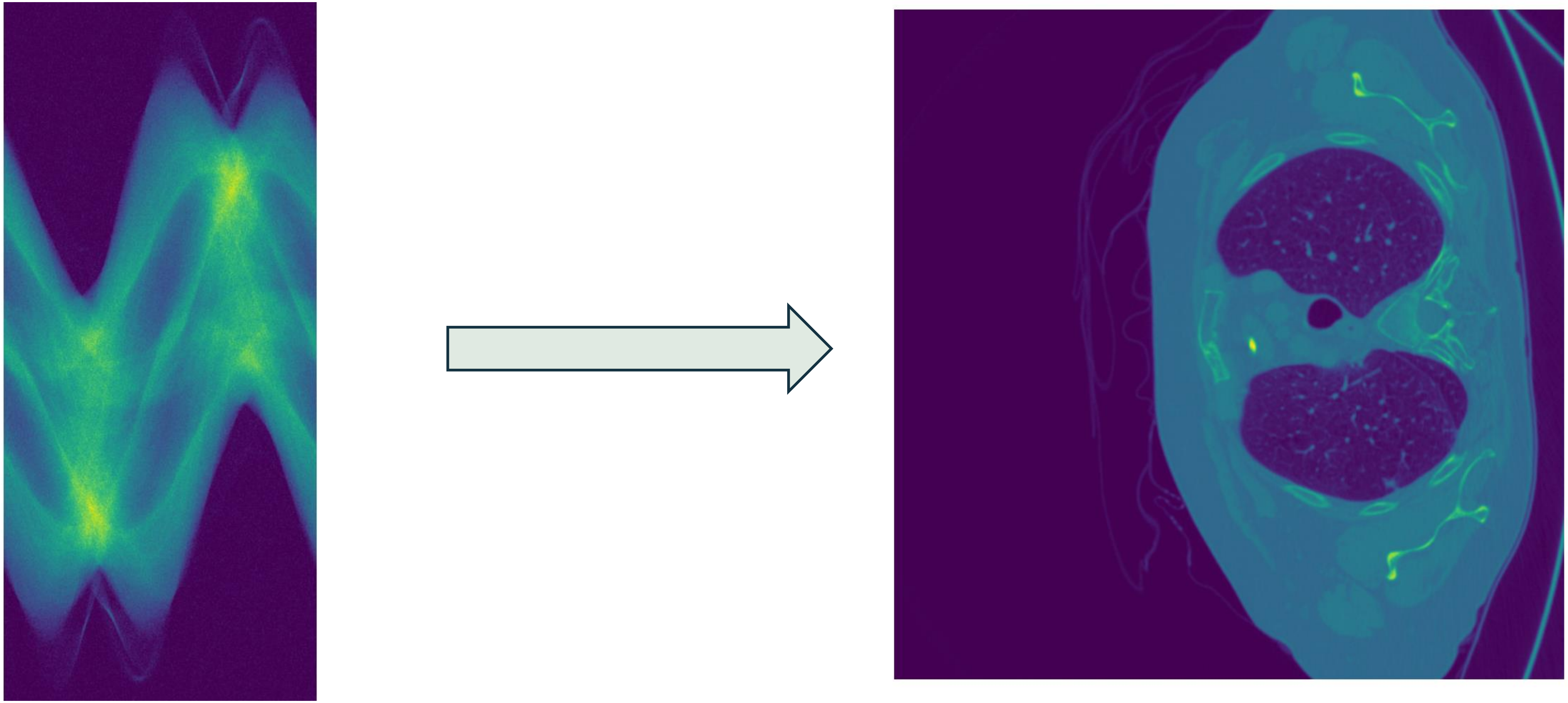


LIDC-IDRI data used in this presentation

# Motivation – Computer Tomography



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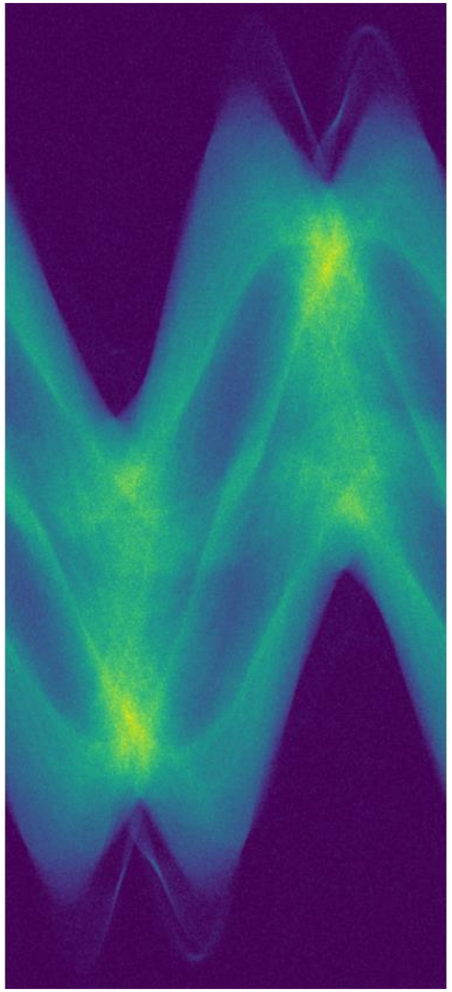
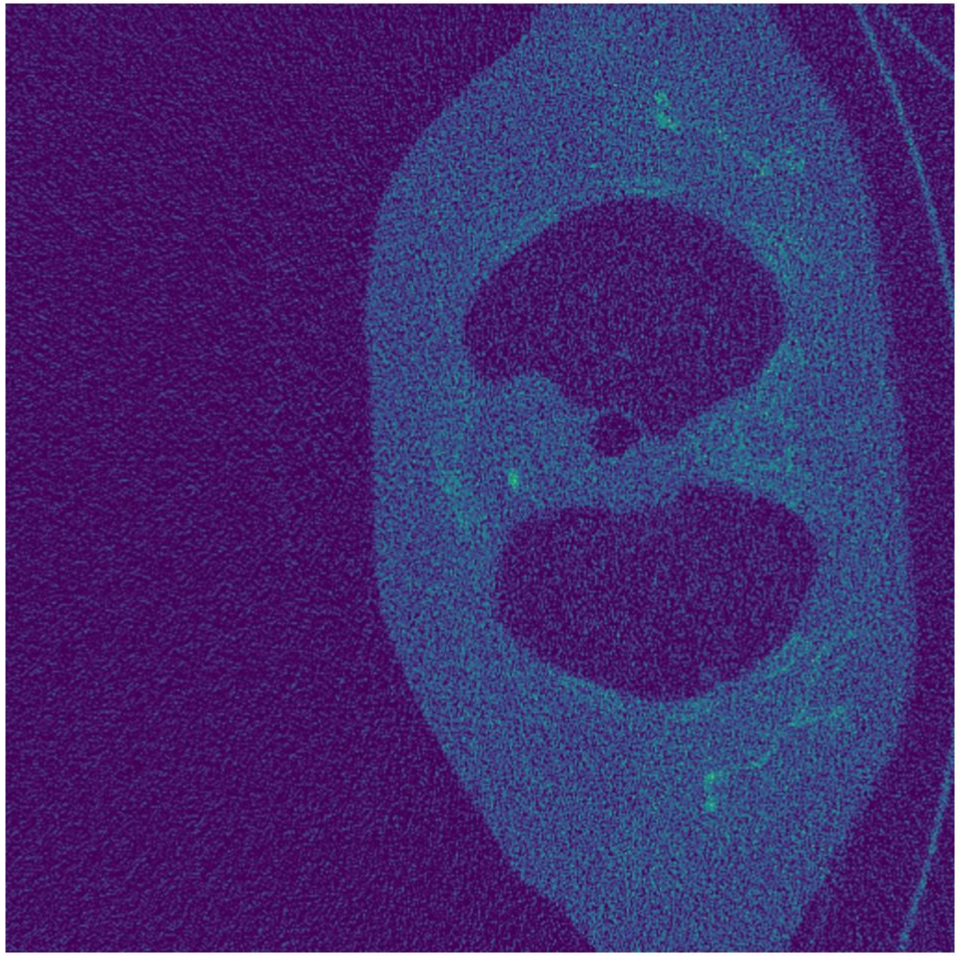




# What do we do with the noise?

$$u(t,x) := \min_{z\in\mathbb{R}^n} \left\{ f(z) + \frac{1}{2t}||x-z||^2 \right\}$$

Moreau Envelope

Denoiser e.g. Total Variation

Data fidelity

$$\text{prox}_{tf}(x) := \underset{\omega\in\mathbb{R}^n}{\text{argmin}} \left\{ \frac{1}{2t}||x-\omega||_2^2 + f(\omega) \right\}$$

Proximal Operator

$$\text{prox}_{tf}(x) = x - t\nabla u(x,t)$$

# Hopf-Lax Formula

$$u(x,t) = \min_y \{tL(\frac{x-y}{t}) + f(y)\}$$ satisfies $$\begin{cases} u_t + H(\nabla u) = 0 \\ u(x,0) = f(x) \end{cases}$$ where $L = H^*$

For our problem $$u(t,x) := \min_{z \in \mathbb{R}^n} \left\{ f(z) + \frac{1}{2t} ||x - z||^2 \right\}$$

Hopf-Lax formula gives a solution to the PDE

$$\begin{cases} u_t + \frac{1}{2}\|\nabla u\|^2 = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = f & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

# Hopf-Cole Transform and Viscosity Solution

$$\begin{cases} u_t^\delta + \frac{1}{2}\|\nabla u^\delta\|^2 = \frac{\delta}{2}\Delta u^\delta & \text{in } \mathbb{R}^n \times (0, T] \\ u^\delta = f & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

and we hope $u^\delta \to u$

Using the transformation $v^\delta \triangleq \exp(-u^\delta/\delta)$,

$$\begin{cases} v_t^\delta - \frac{\delta}{2}\Delta v^\delta = 0 & \text{in } \mathbb{R}^n \times (0, T] \\ v^\delta = \exp(-f/\delta) & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

And we have,

$$\begin{aligned} v^\delta(x, t) &= \Big(\Phi_{\delta t} * \exp(-f/\delta)\Big)(x) \\ &= \mathbb{E}_{y \sim \mathcal{N}(x, \delta t)}\left[\exp\left(-f(y)/\delta\right)\right] \end{aligned}$$

# Viscosity Solution (cont'd)

$$\nabla u^\delta(x,t) = \frac{1}{t}\cdot\left(x - \frac{\mathbb{E}_{y\sim\mathcal{N}(x,\delta t)}\left[y\cdot\exp\left(-f(y)/\delta\right)\right]}{\mathbb{E}_{y\sim\mathcal{N}(x,\delta t)}\left[\exp\left(-f(y)/\delta\right)\right]}\right)$$

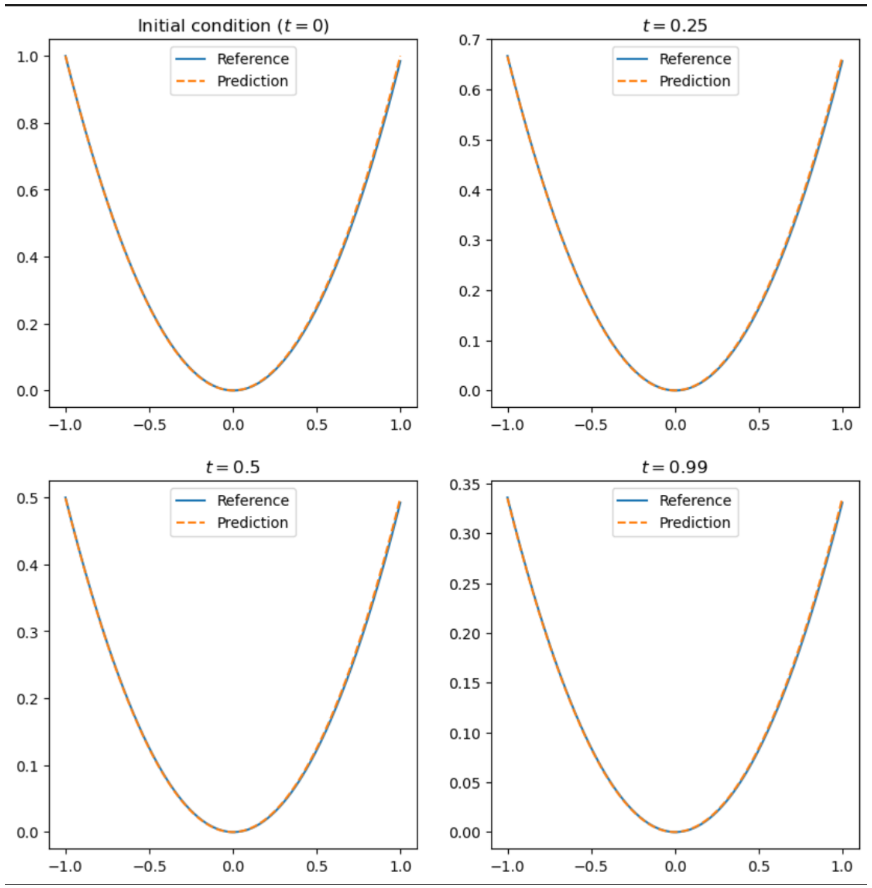
$$\begin{aligned}\operatorname{prox}_{tf}(x) &= x - t\nabla u(x,t)\\ &\approx x - t\nabla u^\delta(x,t)\\ &= \frac{\mathbb{E}_{y\sim\mathcal{N}(x,\delta t)}\left[y\cdot\exp\left(-f(y)/\delta\right)\right]}{\mathbb{E}_{y\sim\mathcal{N}(x,\delta t)}\left[\exp\left(-f(y)/\delta\right)\right]}\end{aligned}$$

# Physics Informed Neural Networks (PINNs)

$$\begin{cases} u_t + \frac{1}{2}\|\nabla u\|^2 = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = f & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Let solution u(x,t) = nn(x,t)

Loss function= $nn_t + \frac{1}{2}\|\nabla nn\|^2$



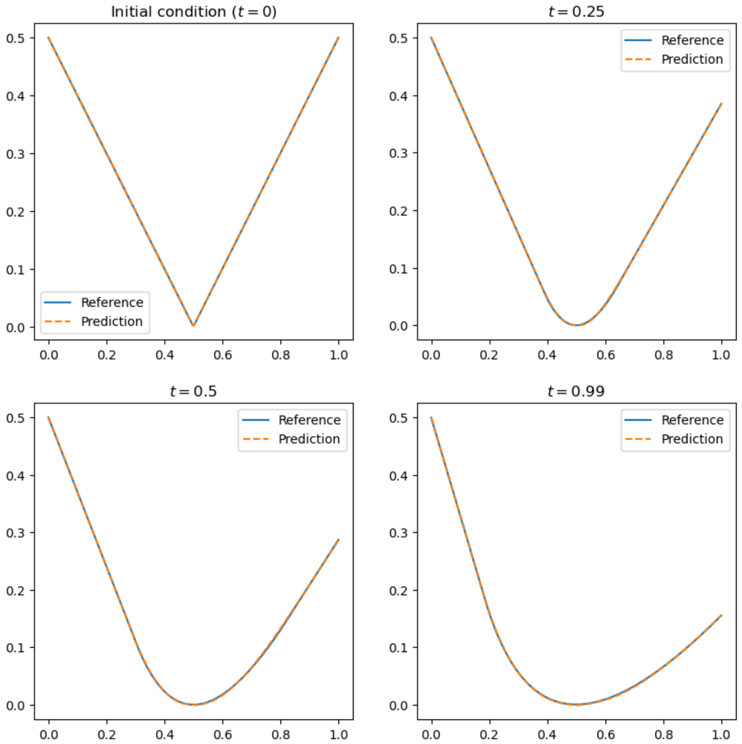

And many more!

# Physics Informed Neural Networks (PINNs)

$$u(x,t) = \min_z g(z) + \frac{1}{t} D_\psi(z,x)$$ with $$\psi(x) = x\log x - x$$

$$\begin{cases} u_t + \frac{1}{t^2} KL(x - xt\nabla u, x) = 0 \\ u(x,0) = |x - \frac{1}{2}| \end{cases}$$

$$u_t + \frac{1}{t^2}\left((x - xt\nabla u)\log(1 - t\nabla u) + tx\nabla u\right) = 0$$



# Thank you!