

M. PHIL. IN STATISTICAL SCIENCE

Monday 11 June 2007 9.00 to 12.00

INTERACTING PARTICLE SYSTEMS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

*Cover sheet
Treasury Tag
Script paper*

SPECIAL REQUIREMENTS

None

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 Let $G = (V, E)$ be a finite connected graph with $|E| \geq 1$, considered as an electrical network with strictly positive conductances w_e , $e \in E$, and source s , sink t . State Kirchhoff's first and second laws for the currents and potential differences of the network. State Ohm's law.

Let $X = (X_n : n \geq 0)$ be a Markov chain on the state space V with transition matrix

$$p_{xy} = \frac{w_e}{\sum_{f \sim x} w_f}$$

where e is the edge $\langle x, y \rangle$ and the summation is over all edges f incident to x . Thus $p_{xy} = 0$ if either $x = y$ or x is not a neighbour of y . The chain starts at $X_0 = s$, and it stops at the first time it visits t .

Let u_{xy} be the expected total number of one-step transitions of the chain from x to y ; each transition from x to y counts $+1$, and from y to x counts -1 . Show that u satisfies the two Kirchhoff laws with a total flow of one, and deduce that u_{xy} equals the current along $\langle x, y \rangle$ from x to y when the total flow equals one.

[A clear statement should be given of any general result to which you appeal.]

2 Let $\Omega = \{0, 1\}^E$ where E is a finite set. Define an *increasing* subset of Ω . For increasing subsets A, B of Ω , define the subset $A \circ B$ [sometimes written $A \square B$] containing vectors $\omega \in \Omega$ for which ' A and B occur disjointly'.

State the BK 'disjoint-occurrence' inequality for the product measure P_p on Ω with density p .

Consider bond percolation on \mathbb{Z}^2 with density p , and let A be the event that there exists an open path that crosses the rectangle $[0, 2n] \times [0, 2n - 1]$ from its left side to its right side. Show that

$$P_p(A) + P_{1-p}(A) = 1.$$

By considering the open clusters at vertices of the form (n, y) for $0 \leq y \leq 2n$, show that

$$P_{\frac{1}{2}}(\text{rad}(C) \geq n) \geq \frac{1}{2\sqrt{n}}$$

where C is the cluster at the origin 0 and $\text{rad}(C) = \max\{n : 0 \leftrightarrow \partial\Lambda_n\}$ with $\Lambda_n = [-n, n]^2$.

3 Let $G = (V, E)$ be a finite graph. Let $p \in (0, 1)$, $q \in \{2, 3, \dots\}$, and write $\Omega = \{0, 1\}^E$ and $\Sigma = \{1, 2, \dots, q\}^V$. Let κ be the probability measure on $\Omega \times \Sigma$ given by

$$\kappa(\omega, \sigma) = \frac{1}{Z} \prod_{e \in E} \left\{ (1-p)\delta_{\omega(e),0} + p\delta_e(\sigma)\delta_{\omega(e),1} \right\},$$

where $\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y}$ for $e = \langle x, y \rangle \in E$.

Show that the first marginal measure of κ is the random-cluster measure $\phi_{p,q}$, and the second marginal measure is the Potts measure $\pi_{\beta,q}$, where $p = 1 - e^{-\beta}$. Derive the conditional measure on Ω given the vertex-configuration σ , and the conditional measure on Σ given the edge-configuration ω .

Prove that

$$\left(1 - \frac{1}{q}\right) \phi_{p,q}(x \leftrightarrow y) = \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q},$$

and explain how this can be used to relate the phase transitions of the random-cluster and the Potts models.

4 Let $\Omega = \{0, 1\}^E$ where E is a finite set, and let μ_1 and μ_2 be probability measures on Ω . Explain what is meant by saying that μ_1 dominates μ_2 stochastically. State the Holley condition for this to occur.

Let $G = (V, E)$ be a finite graph, and let $\phi_{p,q}$ be the random-cluster measure on G with parameters p and q . Prove that

$$\phi_{p',1} \leq_{\text{st}} \phi_{p,q} \leq_{\text{st}} \phi_{p,1}, \quad q \geq 1, \quad p \in (0, 1),$$

where $p' = p/[p + q(1-p)]$ and \leq_{st} denotes stochastic ordering. [You may need the fact that $k(\omega) + \eta(\omega)$ is a non-decreasing function of ω , where $k(\omega)$ is the number of open clusters of ω and $\eta(\omega)$ is the number of open edges.]

Let $p_c(q)$ denote the critical point of the (wired) random-cluster measure on \mathbb{Z}^d , where $q \geq 1$. Show that $p_c(1) \leq p_c(q)$, and derive an upper bound for $p_c(q)$ in terms of $p_c(1)$.

5 Describe the graphical representation of the contact model on \mathbb{Z}^d , in terms of Poisson processes of deaths and infection with respective intensities δ and λ . Let $\delta = 1$, and write $\theta(\lambda)$ for the probability that an initial infection at the origin persists in \mathbb{Z}^d for all future time. Explain why θ is a non-decreasing function, and define the critical point $\lambda_c = \lambda_c(d)$ of the process.

Prove that $\lambda_c \geq (2d)^{-1}$.

By coupling the d -dimensional process with the one-dimensional process with infection rate $d\lambda$, or otherwise, show that $\lambda_c(d) \leq d^{-1}\lambda_c(1)$.

6 Let $G = (V, E)$ be a finite graph, and $\Sigma = \{-1, +1\}^V$. For $x \in V$, $\sigma \in \Sigma$, let $N(x) = \sum_{y \sim x} \sigma_y$ be the sum of the states of the neighbours of x . Consider a discrete-time Markov chain $X = (X_n : n \geq 0)$ on Σ with transition probabilities

$$p(\sigma_x, \sigma^x) = \frac{1}{|V|} \cdot \frac{e^{2N(x)}}{1 + e^{2N(x)}},$$

$$p(\sigma^x, \sigma_x) = \frac{1}{|V|} \cdot \frac{1}{1 + e^{2N(x)}},$$

for $x \in V$, $\sigma \in \Sigma$, where σ_x (respectively, σ^x) is the configuration obtained from σ by setting the value -1 (respectively, $+1$) at the position labelled x . Let $p(\sigma, \sigma') = 0$ if σ, σ' differ at more than one vertex. Show that X is a (time-)reversible Markov chain with respect to the Ising measure

$$\pi(\sigma) = \frac{1}{Z} \exp\left(\sum_{x \sim y} \sigma_x \sigma_y\right), \quad \sigma \in \Sigma,$$

where the summation is over all unordered pairs of neighbouring vertices.

Explain how the chain X may be used in a system of ‘coupling from the past’ in order to generate a random sample with measure π . Prove that the ‘coupling from the past’ algorithm terminates in finite time, with probability one.

END OF PAPER