Algebraic Geometry Exercises

Jack Smith
jes98@cam.ac.uk

1. Draw pictures of $Z(T) \subset \mathbb{A}^2$ for the following choices of T:
   (a) $T = \{Y - X^2\}$.
   (b) $T = \{Y^2 - X^2\}$.
   (c) $T = \{Y - X^2, Y^2 - X^2\}$.
2. Does your picture for 1(c) show every point of the set $Z(T)$?

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3. Given a subset T of $A = k[X_1, \ldots, X_n]$, show that there is a finite subset $S \subset T$ such that $Z(S) = Z(T)$. (Hint: since A is Noetherian, the ideal J generated by T is finitely generated, so there exist polynomials $f_1, \ldots, f_m$ such that $J = (f_1, \ldots, f_m)$. Now write each f_i in terms of some elements of T.)

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4. Prove that $\bigcap_{\alpha \in \mathcal{A}} Z(J_\alpha) = Z\big(\sum_{\alpha \in \mathcal{A}} J_\alpha\big)$ for ideals J_α.
5. For $k = \mathbb{C}$ find a continuous map $\mathbb{A}^1 \to \mathbb{A}^1$ which is not a regular function.

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6. Given a set $Y \subset \mathbb{A}^n$, show that $Z(I(Y))$ is the closure of Y in the Zariski topology.
7. Find an ideal $J \subset k[X, Y]$ such that the coordinate ring of the affine variety $Z(J) \subset \mathbb{A}^2$ is not isomorphic to $k[X, Y]/J$.

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8. If X is an irreducible topological space, and Y is a non-empty open subset of X, show that Y is irreducible.
9. Show that affine space $\mathbb{A}^n$ is compact in the Zariski topology, i.e. that every open cover has a finite subcover. (Hint: assume not, build a strictly descending sequence of closed sets, and apply the argument used to show that decomposition into irreducible components terminates.)

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10. Show that the parabola $Z(Y - X^2) \subset \mathbb{A}^2$ is irreducible.
11. Recall that the closed subsets of $\mathbb{A}^1$ are finite sets of points. Show that such a set is irreducible if and only if it consists of a single point. Directly prove that Z and I induce a correspondence between closed sets and radical ideals in this case, and show that the irreducible closed sets are precisely those corresponding to prime ideals. (Hint: $k[X_1]$ is a principal ideal domain and k is algebraically closed.) The fact that irreducible closed sets in $\mathbb{A}^1$ are single points reflects the fact that prime ideals in $k[X]$ are maximal.
12. If R is a ring, and J an ideal in R, show that the radical of J is also an ideal. (Hint: if $r_1^{n_1} \in J$ and $r_2^{n_2} \in J$ then expand $(r_1 + r_2)^{n_1+n_2}$.)

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13. Find the k-algebra homomorphism ϕ^* corresponding to each of the following morphisms of affine varieties:

(a) $\phi : \mathbb{A}^n \to \mathbb{A}^m$, for $n < m$, given by inclusion as the first n components.

(b) $\phi : \mathbb{A}^n \to \mathbb{A}^m$, for $m < n$, given by projection onto the first m components.

(c) $\phi : Z(XY - 1) \subset \mathbb{A}^2 \to \mathbb{A}^1$ given by projecting the hyperbola onto the X-axis.

Extra Questions

14. Prove the Hilbert basis theorem as follows. Suppopse R is a Noetherian ring, and let I be an ideal in $R[X]$. For each non-negative integer n, let I_n be the set of leading coefficients of degree n polynomials in I; explicitly

$$I_n := \{r \in R : rX^n + r_{n-1}X^{n-1} + \cdots + r_0 \in I \text{ for some } r_0, \ldots, r_{n-1} \in R\} \cup \{0\}.$$

Check that $I_0 \subset I_1 \subset I_2 \subset \ldots$ is an increasing sequence of ideals in R. Deduce that there exists N such that $I_n = I_N$ for all $n \geq N$.

Show that for each n there exist finitely many degree n polynomials $f_{n,1}, \ldots, f_{n,k_n}$ whose leading coefficients generate I_n. Show that the $f_{i,j}$ for $i \leq N$ generate I, by taking a polynomial $f \in I$ and showing by induction on the degree of f that it can be written in terms of these $f_{i,j}$.

15. Show that the decomposition of an affine variety into irreducible components is essentially unique, i.e. that if an affine variety V can be written as $V_1 \cup \cdots \cup V_m$, with each V_i closed and irreducible and not contained in any other V_j, and also as $V_1' \cup \cdots \cup V_{m'}'$ similarly, then $m' = m$ and, reordering the V_i' if necessary, we have $V_i' = V_i$ for all i. (Hint: consider $V_i \cap V_j'$.)

16. Let X and Y be topological spaces, and Z a subset of X.

(a) Show that Z is irreducible if and only if for any pair of open sets $U, V \subset X$ meeting Z, their intersection meets Z (i.e. if $U \cap Z \neq \emptyset$ and $V \cap Z \neq \emptyset$ then $U \cap V \cap Z \neq \emptyset$).

(b) If $f : X \to Y$ is continuous and Z is irreducible, show that $f(Z)$ is an irreducible subset of Y.

17. (a) We've seen the correspondence between points on an affine variety Y and maximal ideals in its coordinate ring $A(Y)$. Let $\operatorname{MaxSpec} A(Y)$ denote the set of maximal ideals of $A(Y)$. Under the correspondence $Y \leftrightarrow \operatorname{MaxSpec} A(Y)$, the Zariski topology on Y induces a topology on $\operatorname{MaxSpec} A(Y)$; describe this topology explicitly. (What do the closed sets look like?)

(b) Now let R be an arbitrary ring (commutative with 1). Let $\operatorname{Spec} R$ denote the set of prime ideals of R. In analogy with your answer to the previous part, define a Zariski topology on $\operatorname{Spec} R$.