Set Theory

The following multiple choice self-test questions are intended to give an indication whether you have prerequisites for the Part III courses in set theory. All of them are basic and easy questions in set theory: you should get most of them right without effort or preparation.

Basics.

1. In the usual formalization of natural numbers and ordered pairs (i.e., $n = \{0, ..., n-1\}$ and $(x, y) := \{\{x\}, \{x, y\}\}$), one of the following statements is true. Which one?

   ☐ **A** $17 \in 4$.

   ☐ **B** $(0, 1) = 2$.

   ☐ **C** $2 \in (0, 1)$.

   ☐ **D** $4 \in (4, 17)$.

2. We often use informal mathematical notation using curly braces to denote sets. However, not every expression corresponds to a set; sometimes, we denote proper classes. Among the following expressions, one corresponds to a proper class. Which one?

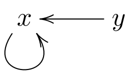
   ☐ **A** $\{x\ ;\ x$ is a nonempty subset of the natural numbers$\}$.

   ☐ **B** $\{x\ ;\ x$ is a finite set of real numbers$\}$.

   ☐ **C** $\{x\ ;\ x$ is a one-element set of rational numbers$\}$.

   ☐ **D** $\{x\ ;\ x$ is a two-element subset of a vector space$\}$.

3. Consider the following model $M = (\{x, y\}, \in)$ as a model of set theory (where $x \in x$ and $x \in y$, but not $y \in x$ or $y \in y$):

   One of the following axiom (scheme)s of set theory is true in M. Which one?

   ☐ **A** The Pairing Axiom.

   ☐ **B** The Axiom of Foundation.

   ☐ **C** The Union Axiom.

   ☐ **D** The Axiom Scheme of Separation.

4. The Zermelo numbers are defined by the following recursion: $z_0 := \varnothing$ and $z_{n+1} := \{z_n\}$. The von Neumann numbers are defined by: $v_0 := \varnothing$ and $v_{n+1} := v_n \cup \{v_n\}$ (i.e., $v_n = n$). One of the following statements is true. Which one?

   ☐ **A** $z_2 = v_2$.

   ☐ **B** $z_2 \in v_4$.

   ☐ **C** $z_2 \subseteq v_2$.

   ☐ **D** $z_2 \in z_4$.

5. One of the following statements about (X, R) implies that (X, R) is a wellorder. Which one?

   ☐ **A** R is a transitive relation.
   ☐ **B** R is a linear relation.
   ☐ **C** X has an R-minimal element.
   ☐ **D** There is a wellorder (Y, S) and an injective function $f : X \to Y$ such that for all $x_0, x_1 \in X$, we have $x_0 \, R \, x_1$ if and only if $f(x_0) \, S \, f(x_1)$.

6. Consider the integers $\mathbb{Z}$ with their natural order $<$ and their natural multiplication $\cdot$. One of the following sets is wellordered by $<$. Which one?

   ☐ **A** $\mathbb{Z}\backslash\{0\}$,
   ☐ **B** $\{z \in \mathbb{Z}\,;\, \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,
   ☐ **C** $\{z \in \mathbb{Z}\,;\, z < 0 \wedge \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,
   ☐ **D** $\{z \in \mathbb{Z}\,;\, \exists x \in \mathbb{Z}(z = x \cdot x)\}$.

Ordinals.

7. One of the following is provable in ZF. Which one?

   ☐ **A** Every transitive set is an ordinal.
   ☐ **B** Every transitive set of ordinals is an ordinal.
   ☐ **C** Every set of ordinals is transitive.
   ☐ **D** None of the above.

8. Only one of the following statements is correct. Which one?

   ☐ **A** There are two different order isomorphisms between ω_1 and ω_1.
   ☐ **B** There are two different order isomorphisms between ω and ω.
   ☐ **C** There are two different order-preserving embeddings from ω to ω_1.
   ☐ **D** There are two different order-preserving embeddings from ω_1 to ω.

9. One of the following ordinal inequalities is true. Which one?

   ☐ **A** $5 \cdot \omega < \omega \cdot 5$.
   ☐ **B** $\omega \cdot 5 < 5 \cdot \omega + 5$.
   ☐ **C** $5 + \omega + \omega \cdot \omega \cdot \omega < \omega + 5 + \omega \cdot \omega \cdot \omega$.
   ☐ **D** $5 \cdot (20 + \omega_1) < 5 \cdot \omega_1$.

Cardinals.

10. The statement "there are no cardinal numbers between $\aleph_0$ and $\aleph_1$" is...

   ☐ **A** ...provable in ZF,

   ☐ **B** ...provable in ZFC, but not in ZF,

   ☐ **C** ...equivalent to the Continuum Hypothesis in the base theory ZFC.

   ☐ **D** None of the above.

11. In ZFC, one of the following statements is equivalent to the continuum hypothesis. Which one?

   ☐ **A** There is a bijection between the power set of $\mathbb{N}$ and the real numbers.

   ☐ **B** Every set of real numbers has an uncountable subset.

   ☐ **C** Every uncountable set of real numbers has a countable subset.

   ☐ **D** Every uncountable set of real numbers has a subset that is in bijection with the real numbers.

12. Work in ZF and consider $W := \{(A, R)\, ;\, A \subseteq \mathbb{N} \text{ and } (A, R) \text{ is a wellorder}\}$. What is the cardinality of the set W?

   ☐ **A** ZF proves that it is $\aleph_0$.

   ☐ **B** ZF proves that it is $\aleph_1$.

   ☐ **C** ZF does not prove that it is $\aleph_1$, but ZFC proves that it is $\aleph_1$.

   ☐ **D** ZFC does not prove that it is $\aleph_1$, but ZFC + CH proves that it is $\aleph_1$.

13. Only one of the following statements is false. Which one?

   ☐ **A** There is an order preserving injection from ω to $\aleph_\omega$.

   ☐ **B** There is an order preserving injection from ω_1 to $\aleph_\omega$.

   ☐ **C** There is a cofinal order preserving injection from ω to $\aleph_\omega$.

   ☐ **D** There is a cofinal order preserving injection from ω_1 to $\aleph_\omega$.

14. Let κ be a cardinal with $\mathrm{cf}(\kappa) = \aleph_0$. Only one of the following statements is provable. Which one?

   ☐ **A** Every injective function from κ to κ has countable range.

   ☐ **B** The set κ is a countable union of sets of cardinality strictly smaller than κ.

   ☐ **C** The cardinal κ is countable.

   ☐ **D** If $\kappa = \aleph_\xi$, then ξ is a countable ordinal.