UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 9 to 11

PAPER 12

RANDOM GRAPHS

Attempt **TWO** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let $k \geq 2$ be fixed and $\omega(n) \to \infty$. Show that if

$$\omega(n)n^{-k/(k-1)} \leq p(n) \leq \frac{\log n + (k-1)\log\log n - \omega(n)}{kn}$$

then a.e. $G_{n,p}$ has a component of order k.

(ii) Let $3 \leq r = r(n) \leq n^{1/3}$ and $0 < p = p(n) < 1$ be such that

$$\binom{n}{r} p^{\binom{r}{2}} \to \infty \quad \text{and} \quad \binom{n}{r+1} p^{\binom{r+1}{2}} \to 0.$$

Prove that a.e. $G_{n,p}$ has clique number r.

2 Show that whp the hitting time of connectedness of a random graph process is precisely the hitting time of having no isolated vertices.

Deduce that if $\alpha : \mathbb{N} \to \mathbb{R}$ is a bounded function then for $p = p(n) = (\log n + \alpha(n))/n$ we have

$$\mathbb{P}(G_{n,p} \text{ is connected})/(1 - e^{-e^{-\alpha}}) \to 1.$$

3 (i) Let Δ be a fixed natural number and, for each $n \geq 2$, let $\mathbf{d} = (d_i)_1^n$ be a sequence of integers such that $\Delta \geqslant d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n \geqslant 1$, the sum $\sum_{i=1}^{n} d_i = 2m$ is even and $2m - n \to \infty$. Also, for each $n \geq 2$, let G_0 be a graph of maximal degree at most Δ with vertex set $V = [n]$.

Denote by $\mathcal{L}(\mathbf{d}; G_0)$ the set of graphs with vertex set V which do not share an edge with G_0 and whose degree sequence is exactly $(d_i)_1^n : d(i) = d_i, i = 1, \ldots, n$. Sketch a proof of the fact that, as $n \to \infty$,

$$|\mathcal{L}(\mathbf{d}; G_0)| \sim e^{-\lambda/2 - \lambda^2/4 - \mu} (2m)_m \Bigg/ \left(2^m \prod_{i=1}^{n} d_i!\right),$$

where

$$\lambda = \frac{1}{m} \sum_{i=1}^{n} \binom{d_i}{2} \quad \text{and} \quad \mu = \frac{1}{2m} \sum_{ij \in E(G_0)} d_i d_j.$$

(ii) Show that, for $\ell, r \geq 3$ fixed and rn even, the expected number of ℓ-cycles in a random r-regular graph $G_{n,r-\mathrm{reg}}$ tends to $(r-1)^\ell/2\ell$ as $n \to \infty$.

4 (i) Let $C(n, n+\ell)$ be the number of connected graphs on $[n]$ with $n+\ell$ edges. Prove that

$$C(n,n) = \frac{1}{2}(n-1)! \sum_{j=0}^{n-3} \frac{n^j}{j!}.$$

(ii) By making use of the 2-core and kernel of a connected graph of order n and size $n+1$, prove that

$$c_1 n^{n+1} \le C(n, n+1) \le c_2 n^{n+1}$$

for some positive constants c_1, c_2 and all $n \ge 3$.