UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2003 9 to 12

PAPER 24

MODULAR FORMS

Attempt **FOUR** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the spaces M_k, S_k of modular and cusp forms of weight k for the modular group $SL_2(\mathbb{Z})$. Show that multiplication by $\Delta = (E_4^3 - E_6^2)/1728$ is an isomorphism between M_k and S_{k+12}, and obtain the dimension formula for even $k \geq 0$:

$$\dim M_k = \begin{cases} \left[\dfrac{k}{12}\right] & \text{if } k \equiv 2 \pmod{12} \\ \left[\dfrac{k}{12} + 1\right] & \text{otherwise.} \end{cases}$$

Show also that every modular form on $SL_2(\mathbb{Z})$ can be expressed as a polynomial in E_4 and E_6.

[You may assume the formula

$$\sum_{\tau \neq i,\rho} v_\tau(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + v_\infty(f) = \frac{k}{12}$$

for non-zero $f \in M_k$.]

ii) Show that if $E_4 \Delta = \sum_{n \geq 1} c_n q^n$ then $c_n \equiv \sigma_{15}(n) \pmod{3617}$.

[The q-expansion of E_{16} is $1 + \dfrac{255}{8 \times 3617} \displaystyle\sum_{n=1}^{\infty} \sigma_{15}(n) q^n$.]

2 Write an account of the theory of Hecke operators for modular forms on $SL_2(\mathbb{Z})$.

3 (i) Let $g(\tau) = \sum_{n\geq 1} b_n q^n$ where $|b_n| << n^\sigma$ for some $\sigma \in \mathbb{R}$. Show that the Dirichlet series $L(g,s) = \sum_{n\geq 1} b_n n^{-s}$ satisfies the Mellin transform formula

$$(2\pi)^{-s}\Gamma(s)L(g,s) = \int_0^\infty g(iy)y^s\,\frac{dy}{y}$$

for $\mathrm{Re}(s)$ sufficiently large.

(ii) Let $f \in S_k(SL_2(\mathbb{Z}))$ and $N \geq 1$, a, d integers with $ad \equiv 1 \pmod N$. By considering a suitable matrix $\begin{pmatrix} a & b \\ N & d \end{pmatrix} \in SL_2(\mathbb{Z})$, show that

$$f\Big(\frac{-1}{N^2\tau} + \frac{a}{N}\Big) = (N\tau)^k f\Big(\tau - \frac{d}{N}\Big).$$

(iii) Suppose further that f has Fourier expansion $\displaystyle\sum_{n\geq 1} c_n q^n$. By considering the Mellin transform of $g(\tau) = f(a/N + \tau)$, show that the function

$$M(f,a/N,s) = \Big(\frac{N}{2\pi}\Big)^s \Gamma(s) \sum_{n=1}^\infty e^{2\pi i a n/N} c_n n^{-s}$$

has the integral representation

$$M(f,a/N,s) = \int_{1/N}^\infty \Big(f\Big(\frac{a}{N} + iy\Big)(Ny)^s + (-1)^{k/2} f\Big(\frac{-d}{N} + iy\Big)(Ny)^{k-s}\Big)\,\frac{dy}{y}$$

and deduce that $M(f,a/N,s)$ has an analytic continuation to $\mathbb{C}$ which satisfies the functional equation

$$M(f,a/N,k-s) = (-1)^{k/2} M(f,-d/N,s).$$

4 (i) State and prove the Poisson summation formula, and use it to show that the theta function

$$\vartheta_{00}(z,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2} t^n \qquad (q = e^{2\pi i\tau},\ t = e^{2\pi i z})$$

satisfies

$$\vartheta_{00}(z/\tau, -1/\tau) = \left(\frac{\tau}{i}\right)^{1/2} e^{\pi i z^2/\tau} \vartheta_{00}(z,\tau)$$

(ii) Show that

$$\frac{\vartheta_{00}(z,\tau)\vartheta_{01}(z,\tau)\vartheta_{10}(z,\tau)\vartheta_{11}(z,\tau)}{\vartheta_{11}(2z,\tau)} = \frac{1}{2}\ \vartheta_{00}(0,\tau)\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)$$

where $\vartheta_{\alpha\beta}(z,\tau) = i^{\alpha\beta} q^{\alpha/8} t^{\alpha/2} \vartheta_{00}(z + \frac{\alpha\tau+\beta}{2}, \tau)$.

[The transformation formula for $\vartheta_{\alpha\beta}$ is

$$(-1)^{\alpha}\vartheta_{\alpha\beta}(z+1,\tau) = \vartheta_{\alpha\beta}(z,\tau) = (-1)^{\beta} q^{1/2} t\, \vartheta_{\alpha\beta}(z+\tau,\tau)\ .\ ]$$

5 i) What does it mean to say that a function on the upper half plane is *modular of weight* k on a subgroup $\Gamma \subset SL_2(\mathbb{Z})$? Show that if $f(\tau)$ is modular of weight k on $SL_2(\mathbb{Z})$ then for any positive integer N, $f(N\tau)$ is modular of weight k on

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}$$

ii) Let $E_2(\tau) = 1 - 24\sum_{n\geq 1} \sigma_1(n) q^n$. Show that

$$E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}.$$

Deduce that $E_2^*(\tau) = E_2(\tau) - \dfrac{3}{\pi y}$ is modular of weight 2 on $SL_2(\mathbb{Z})$, where $y = Im(\tau)$.

[You may assume that Δ has the product expansion $\Delta(\tau) = q\prod(1-q^n)^{24}$, and that $PSL_2(\mathbb{Z})$ is generated by the transformations $\tau \mapsto \tau+1$, $\tau \mapsto -1/\tau$.]

iii) Let $N > 1$ and $c(M) \in \mathbb{C}$ be given for each $M|N$. Show that

$$\sum_{M|N} c(M) E_2(M\tau)$$

is modular of weight 2 on $\Gamma_0(N)$ if and only if $\sum M^{-1} c(M) = 0$.