UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 1.30 to 3.30

PAPER 31

POISSON PROCESSES

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Seeds are planted in a field $S \subset \mathbb{R}^2$. The random way they are sown means that they form a Poisson process on S with density $\lambda(x,y)$. The seeds grow into plants that are later harvested as a crop, and the weight of the plant at (x,y) has mean $m(x,y)$ and variance $v(x,y)$. The weights of different plants are independent random variables. Show that the total weight W of all the plants is a finite random variable with mean

$$\iint_S m(x,y)\lambda(x,y)\; dxdy$$

and variance

$$\iint_S \left\{m(x,y)^2 + v(x,y)\right\} \lambda(x,y)\; dxdy\,,$$

so long as these integrals are finite.

[Any general theorems you use must be clearly stated, but should not be proved.]

2 A Poisson process Π on the interval $S = (-1,1)$ has the density

$$\lambda(x) = (1+x)^{-2}(1-x)^{-3}\,.$$

Show that Π has, with probability 1, infinitely many points in S, and that they can be labelled in ascending order as

$$\ldots X_{-2} < X_{-1} < X_0 < X_1 < X_2 < \ldots$$

with

$$X_0 < 0 < X_1\,.$$

Show that there is an increasing function $f : S \to \mathbb{R}$ with $f(0) = 0$ such that the points $f(X)(X \in \Pi)$ form a Poisson process of unit rate on $\mathbb{R}$, and use the strong law of large numbers to show that, with probability 1,

$$\lim_{n\to+\infty} (2n)^{1/2}\,(1 - X_n) = \tfrac{1}{2}\,.$$

Find a corresponding result as $n \to -\infty$.

3 A line L in $\mathbb{R}^2$ not passing through the origin O can be defined by its perpendicular distance $p > 0$ from O and the angle $\theta \in [0, 2\pi)$ that the perpendicular from O to L makes with the x-axis. Explain carefully what is meant by a Poisson process of such lines L.

A Poisson process of lines L has mean measure μ given by

$$\mu(B) = \iint_B dp\; d\theta$$

for $B \subseteq (0,\infty) \times [0, 2\pi)$. A random countable set $\Phi \subset \mathbb{R}^2$ is defined to consist of all intersections of pairs of lines in Π. Show that the probability that there is at least one point of Φ inside the circle with centre O and radius r is less than

$$1 - (1 + 2\pi r)e^{-2\pi r}\,.$$

Is Φ a Poisson process? Justify your answer.