UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 9 to 12

PAPER 63

STRUCTURE AND EVOLUTION OF STARS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

The notation used is standard and you are reminded of the equations of stellar structure in the form:

$$\frac{dP}{dr} = -\frac{\mathrm{G}m\rho}{r^2};$$

$$\frac{dm}{dr} = 4\pi r^2\rho;$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acr^2T^3};$$

$$\frac{dL_r}{dr} = 4\pi r^2\rho\epsilon;$$

$$P = \frac{\Re\rho T}{\mu} + \frac{1}{3}aT^4.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) A protostar of mass M is fully convective beneath its photosphere which lies at an optical depth $\tau = 2/3$. The stellar material is a mixture of fully ionized hydrogen and helium that behaves like a perfect gas with mean molecular weight μ. Explain why, throughout the star, the pressure $P = KT^{5/2}$, where T is the temperature and K is a constant.

Deduce that the central density and pressure behave as

$$\rho_{\rm c} \propto \frac{M}{R^3} \quad \text{and} \quad P_{\rm c} \propto \frac{M^2}{R^4}$$

and that

$$K(t) = K_0 \mu^{-5/2} M^{-1/2} R^{-3/2}.$$

Assume that a suitable surface boundary condition is $\kappa P = \frac{2}{3} g$, where g is the surface gravity, $\kappa = \kappa_0 \rho T^4$ is the opacity in the atmosphere and radiation pressure can be neglected. Hence deduce that the effective temperature $T_{\rm e}$ obeys

$$T_{\rm e}^8 = \frac{2}{3} G \kappa_0^{-1} K_0^{-2} \Re \mu^4 M^2 R,$$

where $\Re$ is the gas constant.

Given that the gravitational energy $\Omega = -3(5-n)^{-1} G M^2 R^{-1}$ for a polytrope of index n, show that radius evolves with time according to

$$R^{-7/2} - R_0^{-7/2} = \frac{49\pi ac}{6} \left(\frac{2\Re}{3\kappa_0 G} \right)^{\frac{1}{2}} K_0^{-1} \mu^2 M^{-1} (t - t_0),$$

where $R = R_0$ when $t = t_0$.

Deduce that the central temperature rises according to $T_{\rm c} \propto \mu^{11/7} M^{5/7} t^{2/7}$ for $t \gg t_0$.

b) In a Hertzsprung–Russell diagram sketch the paths, with directions, followed by two such protostars of the same mass, one with hydrogen abundance $X_1 = 0.7$ and the other with $X_2 = 0.8$.

Discuss briefly what causes the evolution to deviate from these paths.

c) The equation of state of a partially degenerate gas can be approximated by

$$P = \frac{\Re}{\mu_{\rm i}} \rho T + K_{\rm nr} \rho^{5/3},$$

where $\mu_{\rm i}$ is the mean molecular weight of the ions and $K_{\rm nr}$ is a constant. Assuming that the protostar continues to collapse homologously so that $P_{\rm c} \propto M^2/R^4$ and $\rho_{\rm c} \propto M/R^3$ show that the central temperature reaches a maximum

$$T_{\rm max} \propto M^{4/3} \mu_{\rm i}.$$

What is the implication for low-mass stars?

2 In solar-like stars nuclear burning is dominated by the ppI and ppII chains

$$^{1}\mathrm{H}(^{1}\mathrm{H},\mathrm{e}^{+}\nu)^{2}\mathrm{H}(^{1}\mathrm{H},\gamma)^{3}\mathrm{He}(^{3}\mathrm{He},2^{1}\mathrm{H})^{4}\mathrm{He}$$

$$|$$

$$^{3}\mathrm{He}(^{4}\mathrm{He},\gamma)^{7}\mathrm{Be}(\mathrm{e}^{-},\nu)^{7}\mathrm{Li}(^{1}\mathrm{H},^{4}\mathrm{He})^{4}\mathrm{He}.$$

The reaction rate between species i and j is

$$\frac{\lambda_{ij}n_i n_j}{1+\delta_{ij}}, \qquad (*)$$

where n_i is the number density of species i, δ_{ij} is the Kronecker delta and $\lambda_{ij} \propto \eta^2 \mathrm{e}^{-\eta}$, where $\eta = 42.48(AZ_i^2 Z_j^2 T_6^{-1})^{1/3}$, $A = A_i A_j/(A_i + A_j)$ is the reduced atomic mass of the two reacting nuclei, Z_i is the atomic number of species i and the T_6 is related to temperature T by $T_6 = T/10^6\,\mathrm{K}$. Explain the presence of δ_{ij} in the denominator of (*) and describe briefly the physical processes that give rise to the temperature dependence of λ_{ij}.

The beta decay of ^{7}Be is fast compared to all other reactions so that ^{7}Li is the predominant species of atomic mass 7 and all major species can be identified by $i \approx A_i$. Show that the temperature dependence of the rate r_{11} at the centre of the Sun, where $T_6 \approx 15$, of the reaction $^{1}\mathrm{H}(^{1}\mathrm{H},\mathrm{e}^{+}\nu)^{2}\mathrm{H}$ can be written as $r_{11} \propto T^{\alpha}$, where $\alpha = \frac{1}{3}(\eta - 2) \approx 4$. Also show that β and γ are approximately 16 (with $\gamma > \beta$) in the expressions $r_{33} \propto T^{\beta}$ and $r_{34} \propto T^{\gamma}$.

Show that the rate of change of protons obeys

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = -\lambda_{11}n_1^2 - \lambda_{21}n_2 n_1 + \lambda_{33}n_3^2 - \lambda_{17}n_1 n_7,$$

and obtain the equivalent equations for n_2, n_3 and n_4.

At the centre of the Sun the characteristic timescale of r_{11} is about 10^{10} yr while that of r_{12} is about 1 s. The characteristic timescale for n_3 to reach equilibrium is $\tau \approx 6\times10^5$ yr. By making an appropriate approximation, to be explained, show that

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} \approx -\frac{3}{2}\lambda_{11}n_1^2 + \lambda_{33}n_3^2 - \lambda_{17}n_1 n_7$$

and

$$\frac{\mathrm{d}n_3}{\mathrm{d}t} \approx \frac{1}{2}\lambda_{11}n_1^2 - \lambda_{33}n_3^2 - \lambda_{34}n_3 n_4$$

near the centre of the Sun.

Show further that $n_3 \approx n_{3\mathrm{e}}$ where

$$n_{3\mathrm{e}} = -\frac{\lambda_{34}n_4}{2\lambda_{33}} + \sqrt{\left(\frac{\lambda_{34}n_4}{2\lambda_{33}}\right)^2 + \frac{\lambda_{11}n_1^2}{2\lambda_{33}}}.$$

Consider a small perturbation of the form $n_3 = n_{3\mathrm{e}} + x$ about this equilibrium and linearize the evolution equation for n_3 to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\tau},$$

where $\tau = (2\lambda_{33}n_{3e} + \lambda_{34}n_4)^{-1}$.

Estimate the temperature at which τ is comparable to the age of the Sun.

Sketch the abundances X_1 and X_3 of ^{1}H and ^{3}He as a function of radius in the Sun today.

3 A red giant of mass M_1 is in a binary system with a main-sequence star of mass M_2. The red giant is losing mass in a fast spherically symmetric stellar wind at a rate $\dot{M} < 0$. Show that, if the intrinsic angular momentum of the stars is neglected,

$$\frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} = \frac{M_2\dot{M}}{M_1 M},$$

where $M = M_1 + M_2$ and that the orbital period P and separation a obey

$$P \propto M^{-2}, \quad a \propto M^{-1}$$

On a short timescale the radius of the giant R_1 responds according to

$$R_1 \propto M_1^{-n} \quad 0 < n < 1$$

and the radius of its Roche lobe is approximated by

$$\frac{R_{\text{L}}}{a} = 0.426 \left(\frac{M_1}{M}\right)^{\frac{1}{3}}.$$

Now suppose that the giant is filling its Roche lobe and that wind mass loss is taking place on a timescale much shorter than the nuclear timescale. Show, by differentiating $\log(R_1/R_{\text{L}})$ or otherwise, that mass transfer is driven by the wind if

$$q = \frac{M_1}{M_2} < \frac{1+3n}{3(1-n)} \qquad (\dagger)$$

and that otherwise the wind drives the system to a detached state.

Show further that, when (†) is satisfied and $6q < 5 - 3n$, the rate of mass transfer to the main-sequence star $\dot{M}_2 = \dot{M} - \dot{M}_1$ is given by

$$\dot{M}_2 = -\frac{1 + 3n - 3(1-n)q}{(1+q)(5-3n-6q)}\dot{M}.$$

What is the physical consequence if $6q > 5 - 3n$?

4 Write brief notes on **three** of the following

a) The use of polytropes as stellar models

b) The evidence that stars are powered by nuclear reactions

c) Type Ia supernovae

d) X-ray binary stars.