UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 1.30 to 3.30

PAPER 79

LARGE DEVIATIONS AND QUEUES

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may find helpful the reference material at the end of the paper.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $A_1, A_2, \ldots$ be normal random variables with mean μ and variance σ^2. Let B be an exponential random variable with mean $1/\lambda$. Let C be a normal random variable with mean ν and variance ρ^2. Let all of these random variables be independent.

(a) State, without proof, a large deviations principle for $L^{-1}B$.

(b) Find a large deviations principle for $L^{-1}(A_1 + \cdots + A_L)$.

(c) Find a large deviations principle for $L^{-1}(B + A_1 + \cdots + A_L)$.

(d) Find a large deviations principle for $L^{-1}(C + A_1 + \cdots + A_L)$.

(e) Comment on your results.

State clearly any general results to which you appeal.

2 (a) Define these terms: rate function, good rate function, large deviations principle.

Recall that a sequence of random variables $(X_L, L \in \mathbb{N})$ is said to be *exponentially tight* if for all $\alpha \geqslant 0$ there exists a compact set K_α such that

$$\limsup_{L\to\infty} \frac{1}{L} \log P(X_L \notin K_\alpha) < -\alpha.$$

The sequence $(X_L, L \in \mathbb{N})$ is said to satisfy a *weak large deviations principle* if the large deviations upper bound is required to hold only for compact sets.

Suppose that the sequence $(X_L, L \in \mathbb{N})$ is exponentially tight, and satisfies a weak large deviations principle with rate function I.

(b) Show that I is a good rate function.

(c) Show that the large deviations upper bound holds for closed sets.

Conclude that $(X_L, L \in \mathbb{N})$ satisfies a large deviations principle with good rate function I.

3 (a) Consider a queue operating in slotted time, with infinite buffer and fixed service rate c, and receiving an amount of work a_t in timeslot $(t-1,t)$. What is the Lindley recursion for queue size? Writing a for $(a_t,\ t\in\mathbb{Z})$, define the queue size function $Q_0(a,c)$.

(b) Fix $\lambda > 0$ and consider the space of input process

$$\mathcal{X} = \left\{ a : \lim_{t\to\infty} \frac{a_{-t} + \cdots + a_{-1}}{t} = \lambda \right\}$$

equipped with the norm

$$||a|| = \sup_{t\in\mathbb{N}} \left| \frac{a_{-t} + \cdots + a_{-1}}{t+1} \right|$$

Show that, if $\lambda < c$, the queue size function $Q_0(\cdot, c)$ is continuous on $(\mathcal{X}, ||\cdot||)$.

(c) Suppose that work from this queue is fed into another queue downstream: any work served by the first queue in timeslot $(t-1,t)$ reaches the downstream queue in the same timeslot, and may be served in the same timeslot. Let the downstream queue have service rate $d < c$. Write down a recursion for the downstream queue size $R_t(a,c,d)$, and show that $R_t(a,c,d)$ satisfies

$$Q_t(a,c) + R_t(a,c,d) = \left[Q_{t-1}(a,c) + R_{t-1}(a,c,d) + a_t - d\right]^+.$$

(d) Define the downstream queuesize function $R_0(a,c,d)$.

(e) Suppose that $\lambda < d < c$. Show that $R_0(\cdot, c, d)$ is continuous on $(\mathcal{X}, ||\cdot||)$. Explain how one might use this in finding a large deviations principle for the downstream queuesize.

4 Define the *effective bandwidth* of an arrival process. Write an essay on effective bandwidths and large deviations. In your essay you should describe a queueing model, explain the use of large deviations theory in analysing it, interpret the results in terms of effective bandwidth, and give examples, including an example of a queue fed by several independent arrival processes.

(You should prove a large deviations upper bound for the queue length distribution, but you need not prove a large deviations lower bound.)

Reference: Gärtner-Ellis theorem

A convex function $\Lambda : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is *essentially smooth* if

(a) the interior of its effective domain is non-empty

(b) $\Lambda(\cdot)$ is differentiable throughout the interior of its effective domain

(c) $\Lambda(\cdot)$ is steep, namely, $|\nabla\Lambda(\theta_n)| \to \infty$ whenever (θ_n) is a sequence in the interior of the effective domain converging to a point on the boundary of the effective domain.

Let $(X_L,\ L \in \mathbb{N})$ be a sequence of random vectors in $\mathbb{R}^d$, and let

$$\Lambda^L(\theta) = \frac{1}{L} \log E \exp(L\theta \cdot X_L)$$

for $\theta \in \mathbb{R}^d$. Assume that for each θ the limit

$$\Lambda(\theta) = \lim_{L\to\infty} \Lambda^L(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$. Assume further that 0 is in the interior of the effective domain of Λ, and that Λ is essentially smooth and lower-semicontinuous. Then $(X_L,\ L \in \mathbb{N})$ satisfies an LDP in $\mathbb{R}^d$ with good rate function

$$\Lambda^*(x) = \sup_{\theta\in\mathbb{R}^t} \theta \cdot x - \Lambda(\theta).$$