UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June, 2003 9 to 12

PAPER 9

BOUNDED ANALYTIC FUNCTIONS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with no zeros and $f(0) = c$. Find the maximum value of $|f'(0)|$. Is this maximum attained and, if so, for which functions?

Find the extreme values of $|f(w)|$ for $w \in \mathbb{D}$.

(b) State Harnack's inequality.

Let $u : \mathbb{D} \to (0, \infty)$ be a positive harmonic function. Prove that, for each point $z \in \mathbb{D}$, the gradient ∇u satisfies

$$|\nabla u(z)| \leqslant cu(z)$$

for some constant c that depends on z but not on u. What is the best possible value for c?

2 Prove that a Blaschke product B has

$$\lim_{r \to 1} \int_0^{2\pi} \log |B(re^{i\theta})| \, \frac{d\theta}{2\pi} = 0 \ .$$

Suppose that $f : \mathbb{D} \to \mathbb{C}$ is an analytic function and satisfies

$$\lim_{r \to 1} \int_0^{2\pi} \big|\log |f(re^{i\theta})|\big| \, \frac{d\theta}{2\pi} = 0 \ .$$

Prove the following assertions.

(a) f has bounded characteristic.

(b) $f(z) = B(z)g(z)$ for some Blaschke product B and some analytic function g with no zeros.

(c) f is itself a Blaschke product.

3 Write an essay on the non-tangential limits of a bounded analytic function.

[*You should give a clear outline of a proof that a bounded analytic function has non-tangential limits at almost every point of the unit circle but you are not expected to give complete proofs of every aspect of this result.*]

4 Let (z_n) be a sequence of points in $\mathbb{D}$ with $|z_n| \to 1$ as $n \to \infty$. Show that there is an analytic function on $\mathbb{D}$ with zeros precisely at the points (z_n).

Show that there is a bounded analytic function on $\mathbb{D}$ with zeros precisely at the points (z_n) if and only if

$$\sum e^{-\rho(0,z_n)} < \infty \ .$$

[*You may state the Poisson–Jensen formula without proof.*]

Suppose that (z_n) is the sequence of zeros of a Blaschke product B. Show that

$$|B(w)| \leqslant \exp\left(-2\sum e^{-\rho(w,z_n)}\right)$$

for any point $w \in \mathbb{D}$.

[*You may wish to show first that*

$$\log\frac{1+t}{1-t} \geqslant 2t \qquad \text{for } t \in [0,1)$$

and then set $t = e^{-\rho(w,z_n)}$.]

5 Prove that the following conditions on a positive Borel measure μ on the unit disc $\mathbb{D}$ are equivalent.

(a) There is a constant $N(\mu)$ with

$$\int_{\mathbb{D}} |f| \, d\mu \leqslant N(\mu)||f||_1 \qquad \text{for all} \quad f \in \mathrm{H}_1(\mathbb{D}) \ .$$

(b) There is a constant $C(\mu)$ with $\mu(Q(I)) \leqslant C(\mu)m(I)$ for all intervals I on $\partial\mathbb{D}$.

Here I is an interval on the unit circle and $Q(I)$ is the hyperbolic half-plane bounded by I and the hyperbolic geodesic joining the endpoints of I.