UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Thursday 3 June, 2004 1.30 to 4.30

PAPER 66

GALAXIES AND DARK MATTER

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Laplace's equation in cylindrical coordinates for a disk is

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right)+\frac{\partial^2\Phi}{\partial z^2}=0\ .$$

Considering a separable $\Phi(R,z)=J(R)Z(z)$, show that, identifying appropriate boundary conditions, Laplace's equation is solved by the potential-density pair

$$\Sigma(R)=\frac{-\mathrm{k}}{2\pi G}J_0(\mathrm{k}R)$$

$$\Phi(R,z)=\exp(-\mathrm{k}|z|)J_0(\mathrm{k}R)$$

where Σ is the surface density, and k is to be explained.

Use this to find an expression for the potential associated with a disk of arbitrary surface density $\Sigma(R)$.

Apply your result to show that the circular speed of the Mestel disk,

$\Sigma_m(R)=\Sigma_0 R_0 R^{-1}$, with Σ_0 and R_0 constant, is given by

$$v_c^2(R)=GM(R)R^{-1}\ ,$$

with $M(R)$ the mass interior to R.

Consider the distribution function

$$F(\varepsilon,L_z)=FL_z^q\exp(\varepsilon/\sigma):L_z>0$$

$$F(\varepsilon,L_z)=0;L_z\leqslant 0$$

where ε is energy, L_z angular momentum about the z symmetry axis and σ a constant. Use this to determine q and F such that this is the distribution function of the Mestel disk.

2 Show that the first velocity moment of the collisionless Boltzman equation, for a spherical system, can be written

$$\frac{\mathrm{d}}{\mathrm{dr}}(\lambda\sigma_r^2) + \frac{2\beta}{r}\lambda\sigma_r^2 = -\gamma\frac{\lambda \mathrm{L}(r)}{r^2} \ ,$$

where σ_r is the radial component of the velocity dispersion, $\beta = 1 - (\sigma_t^2/\sigma_r^2)$, with σ_t the tangential component, λ is the luminosity density, $\gamma =$ G times the mass to light ratio, and $\mathrm{L}(r)$ the enclosed luminosity.

The surface brightness μ may be related to λ by

$$\lambda(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}x}{(x^2 - r^2)^{1/2}}\mu'(x) \ .$$

By considering the equivalent relation for the projected velocity dispersion, $\sigma_{\mathrm{p}}^2\mu$, show that the observables can be related to a function of coordinates and $\lambda\sigma_r^2$ by

$$\sigma_{\mathrm{p}}^2\mu - \gamma\int_r^\infty \frac{\mathrm{d}x}{(x^2 - r^2)^{1/2}}\frac{r^2}{x^2}\gamma\mathrm{L} \equiv \mathrm{p}(r) \ .$$

From term by term inspection, show

$\int_0^\infty \ 6\pi r \mathrm{p}(r)\mathrm{d}r = 0$

is equivalent to the virial theorem, so that

$$\gamma = \frac{3\int_0^\infty r\sigma_{\mathrm{p}}^2\mu \mathrm{d}r}{2\int_0^\infty r\lambda \mathrm{L}\mathrm{d}r} \ .$$

3 The observed phase space density of binary stars depends on orbital eccentricity as $F(e) \propto e^2$. Jeans considered the Boltzmann distribution $dN \propto \exp(-E/\sigma)$, N particle number, E energy, to show this predicts $F(e) \propto e^2$, and concluded binary stars are a relaxed system, with age $\sim 10^{13}$ years.

(i) Show the relaxation time for a stellar system like the solar neighbourhood is

$$\mathrm{T}_R = \frac{3}{16\pi\sqrt{2}} \frac{\sigma^3}{\mathrm{N G^2 m^2} ln\Lambda} \ ,$$

where σ is a representative 1-D velocity, m a star mass, N the number of stars, and Λ is to be explained.

(ii) Now consider a phase space distribution of binaries

$$dN = F(E) dx\ dy\ dz\ dp_x\ dp_y\ dp_z \ ,$$

where $x,\ y,\ z$ are spatial coordinates, and p_i the momenta.

Perform a canonical transformation to Delaunay elements, $L,\ G,\ H$, and corresponding angular variables $l,\ g,\ h$ where

$$\begin{aligned} L^2 &= \gamma m a \\ G^2 &= \gamma m a (1 - e^2) \\ H^2 &= \gamma m a (1 - e^2)(\cos^2 i) \end{aligned}$$

where a and e are the semi-major axis of and eccentricity of the orbit, i its inclination and m the system total mass, γ is the Newtonian gravitational constant. Deduce the ranges of the independent orthogonal variables $L,\ G,\ H,\ l,\ g,\ h$.

Show in this case the number of stars with eccentricity less than some value e is $N \propto e^2$, for any finite $F(E)$. Conclude that one cannot deduce equilibrium from an observed $F(e) \propto e^2$.

4 (i) Define an isolating integral of an orbit. Under what conditions are there the following numbers of isolating integrals of an orbit

1) one

2) none

3) more than three

(ii) Show the (Jaffe) potential generated by the spherical density distribution:

$$\rho(r) = \frac{m}{4\pi r_j^3} \frac{r_j^4}{r^2(r + r_j)^2}$$

is

$$\Phi(r) = \frac{Gm}{r_j} ln \left\{ \frac{r}{r + r_j} \right\} \ ,$$

with $M,\ r_j$ constants.

Verify that the total mass is M.

Show the circular speed $v_c(r)$ is approximately constant for $r \ll r_j$, and falls off as:

$$v_c(r) \propto r^{-1/2} \ , \ r \gg r_j \ .$$