UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 8 June, 2004 9 to 11

PAPER 68

ACCRETION DISCS

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) A particle moves freely in an axisymmetric gravitational potential $\Phi(r,z)$, where (r,ϕ,z) are cylindrical polar coordinates. Show that the particle moves in the rz-plane under an effective potential

$$\Phi_1(r,z) = \Phi(r,z) + \frac{h^2}{2r^2},$$

where h is the specific angular momentum. Sketch the contour lines of Φ_1 when Φ is a point-mass potential. Describe the dynamics in the neighbourhood of any stationary points of Φ_1. Explain the significance of this analysis for the stability of circular Keplerian orbits.

(b) The azimuthal motion of the particle is now constrained such that it is forced to rotate with angular velocity ω. Show that the particle now moves under an effective potential

$$\Phi_2(r,z) = \Phi(r,z) - \frac{1}{2}r^2\omega^2.$$

Sketch the contour lines of Φ_2 when Φ is a point-mass potential. Describe the dynamics in the neighbourhood of any stationary points of Φ_2. Explain the significance of this analysis for the magnetic launching of outflows from accretion discs.

(c) An inviscid fluid disc obeys the equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi,$$

where Φ is a point-mass potential. The disc is steady and axisymmetric, in a state of pure differential rotation with $\mathbf{u} = r\Omega(r,z)\,\mathbf{e}_\phi$. Show that, if the pressure and density satisfy a functional relationship of the form $p = K\rho^2$, where K is a constant, then Ω is independent of z and the equilibrium of the disc is given by

$$2K\rho + \Phi_3 = \text{constant},$$

where

$$\Phi_3(r,z) = \Phi(r,z) - \int r\Omega^2\,\mathrm{d}r.$$

If the semi-thickness $H(r)$ of the disc satisfies $H = \epsilon r$, where ϵ is a dimensionless constant, deduce that the angular velocity is given by

$$\Omega^2 = (1+\epsilon^2)^{-1/2}\left(\frac{GM}{r^3}\right),$$

where M is the mass of central object. Show further that the surface density is independent of r.

2 An accretion disc is losing mass and angular momentum through an outflow perpendicular to the plane of the disc. If $S(r,t)$ and $T(r,t)$ are the rates at which mass and angular momentum are lost per unit area, you may assume that the one-dimensional evolutionary equations for mass and angular momentum may be written in the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{2\pi r}\frac{\partial \mathcal{F}}{\partial r} = -S,$$

$$\frac{\partial}{\partial t}(\Sigma h) + \frac{1}{2\pi r}\frac{\partial}{\partial r}(\mathcal{F}h + \mathcal{G}) = -T,$$

where $\Sigma(r,t)$ is the surface density, $\mathcal{F}(r,t)$ is the radial mass flux, $h(r)$ is the specific angular momentum and $\mathcal{G}(r,t)$ is the viscous torque.

(a) Deduce the evolutionary equation for the surface density in the form

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left\{\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1}\left[\frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right) + r(Sh - T)\right]\right\} - S,$$

where $\bar{\nu}(r,t)$ is the effective mean kinematic viscosity and $\Omega(r)$ is the angular velocity. For a Keplerian disc deduce that

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[2(q-1)r^2 S\right] - S, \qquad (*)$$

where q is defined such that $T = qSh$.

(b) When no mass or angular momentum is lost through an outflow, find the viscosity law $\bar{\nu} = \bar{\nu}(r)$ for which equation (*) may be transformed into the classical diffusion equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2}$$

by a suitable change of variables. Explain in physical terms why viscous transport leads to a diffusive redistribution of mass in the disc.

(c) When the disc has no effective viscosity, but loses mass and angular momentum such that $S = \Sigma f(r)$ and $q =$ constant with $q > 1$, show that equation (*) may be transformed into the linear advection equation

$$\frac{\partial g'}{\partial t} = \frac{\partial g'}{\partial x'}$$

by a suitable change of variables. Deduce that the solution takes the form of a wave that propagates inwards through the disc. Describe briefly a physical mechanism by which an outflow satisfying the condition $q > 1$ might occur.

3 In a frame of reference rotating with uniform angular velocity $\boldsymbol{\Omega}$, an incompressible fluid of uniform density ρ, kinematic viscosity ν and magnetic diffusivity η obeys the equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla \Psi + \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \nabla \mathbf{B} + \nu \nabla^2 \mathbf{u}$$

and the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}.$$

State two further conditions satisfied by $\mathbf{u}$ and $\mathbf{B}$, and explain the meaning of the quantity Ψ.

Let (x, y, z) be Cartesian coordinates such that $\boldsymbol{\Omega} = \Omega\, \mathbf{e}_z$, and let A be a constant such that $A(2\Omega - A) > 0$. Show that there exist exact solutions in which

$$\begin{aligned} \mathbf{u} &= -2Ax\, \mathbf{e}_y + \mathrm{Re}\left\{\mathbf{v}(t)\, \exp\left[\mathrm{i}\mathbf{k}(t) \cdot \mathbf{x}\right]\right\}, \\ (\mu_0 \rho)^{-1/2} \mathbf{B} &= \bar{\mathbf{b}}(t) + \mathrm{Re}\left\{\mathbf{b}(t)\, \exp\left[\mathrm{i}\mathbf{k}(t) \cdot \mathbf{x}\right]\right\} \\ \Psi &= -2\Omega A x^2 + \mathrm{Re}\left\{\psi(t)\, \exp\left[\mathrm{i}\mathbf{k}(t) \cdot \mathbf{x}\right]\right\}, \end{aligned}$$

provided that $\bar{\mathbf{b}}(t)$ and $\mathbf{k}(t)$ evolve in time in a specified way, and give the interpretation of this evolution. Verify that the Alfvén frequency $\omega_{\mathrm{A}} = \mathbf{k} \cdot \bar{\mathbf{b}}$ is independent of time, and obtain the evolutionary equations for the wave amplitudes

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - 2Av_x\, \mathbf{e}_y + 2\Omega\, \mathbf{e}_z \times \mathbf{v} = -\mathrm{i}\mathbf{k}\psi + \mathrm{i}\omega_{\mathrm{A}} \mathbf{b} - \nu k^2 \mathbf{v},$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t} = -2Ab_x\, \mathbf{e}_y + \mathrm{i}\omega_{\mathrm{A}} \mathbf{v} - \eta k^2 \mathbf{b},$$

$$\mathbf{k} \cdot \mathbf{v} = \mathbf{k} \cdot \mathbf{b} = 0.$$

Verify that, when the magnetic Prandtl number ν/η is equal to unity, there exists a solution for which $k_x = k_y = 0$, $v_x = v_y$, $v_z = 0$, $b_x = -b_y$ and $b_z = 0$, provided that the Alfvén frequency is chosen appropriately. Determine the condition for the disturbance to grow exponentially, and show that this condition implies a lower limit on the vertical magnetic field strength required in a Keplerian disc. Explain why the properties $v_x = v_y$ and $b_x = -b_y$ are especially conducive to angular momentum transport in an accretion disc.