UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 9 to 12

PAPER 69

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Attempt **THREE** *questions from Section A and attempt* **ONE** *question from Section B.*

There are **seven** *questions in total.*

Each question from Section B carries twice the weight of each question from Section A.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 (a) An s-stage Runge–Kutta method is said to be a *Radau* scheme if it is of order $2s-1$, the matrix A is nonsingular and $\boldsymbol{b}^\top A^{-1}\mathbf{1} = 1$. Prove that such a method is A-stable. [*You may use without proof the theorem that m/n Padé approximants to the exponential lead to A-stable methods for $n-2 \leqslant m \leqslant n$.*]

(b) Identify the Radau scheme with $s = 1$ and show that it can be written as a collocation method.

2 The equation

$$\frac{\partial u}{\partial t} = a(x)\frac{\partial u}{\partial x},$$

where $0 < \alpha \leqslant a(x) \leqslant \beta < \infty$, accompanied by zero boundary conditions, is discretized for $0 \leqslant x \leqslant 1$ by the method

$$u_m^{n+1} = u_m^n + \frac{\Delta t}{(\Delta x)^2}[a_{m-\frac{1}{2}}u_{m-1}^n - (a_{m-\frac{1}{2}} + a_{m+\frac{1}{2}})u_m^n + a_{m+\frac{1}{2}}u_{m+1}^n],$$

where $m = 1, 2, \ldots, M-1$ and $\Delta x = 1/M$.

(a) Express the error as a power of Δx.

(b) Prove that the method is stable, provided that

$$\Delta t \leqslant \frac{(\Delta x)^2}{2\beta}.$$

3 The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

where $-\infty < x < \infty$ and $t \geqslant 0$, given together with an $\mathrm{L}_2[\mathbb{R}]$ initial conditon, is semi-discretized by a method of the form

$$u_m' = \frac{1}{(\Delta x)^2}[\alpha u_m + \beta(u_{m-1} + u_{m+1}) + \gamma(u_{m-2} + u_{m+2})], \qquad m \in \mathbb{Z}.$$

(a) Find values of α, β and γ so that the order of the method, expressed in powers of Δx, is the largest possible.

(b) Determine whether the highest-order method from the Part (a) above is stable.

4 (a) Stating carefully all required conditions, formulate the Lax–Milgram theorem for the solution of Galerkin methods.

(b) Consider the equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)u = f(x), \qquad 0 \leqslant x \leqslant 1,$$

where $u(0) = u(1) = 0$ and there exist constants p_0, p_1, q_1 such that

$$0 < p_0 \leqslant p(x) \leqslant p_1, \qquad 0 \leqslant q(x) \leqslant q_1, \qquad 0 \leqslant x \leqslant 1.$$

Prove that this equation satisfies the conditions of the Lax–Milgram theorem.

5 Given $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$, set

$$\boldsymbol{g} := \frac{\partial \boldsymbol{f}(\boldsymbol{y})}{\partial \boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y})$$

and consider the two-step method

$$\boldsymbol{y}_{n+2} - \frac{7}{15}h\boldsymbol{f}(\boldsymbol{y}_{n+2}) + \frac{1}{15}h^2\boldsymbol{g}(\boldsymbol{y}_{n+2}) = \boldsymbol{y}_n + h\left(\frac{16}{15}\boldsymbol{f}(\boldsymbol{y}_{n+1}) + \frac{7}{15}\boldsymbol{f}(\boldsymbol{y}_n)\right) + \frac{1}{15}h^2\boldsymbol{g}(\boldsymbol{y}_n).$$

(a) Determine the order of this method.

(b) Prove that the linear stability domain of this method is bounded.

[*Hint: Consider the equation* $y' = \lambda y$ *for* $|\lambda| \gg 1$.]

Section B

6 Write a brief essay, accompanied by examples, on convergence of multistep methods for ordinary differential equations. You should describe conditions linking order and convergence, maximal order attainable by a convergent multistep method and the design of convergent methods of high order.

7 Write a brief essay, accompanied by examples, on the eigenvalue method in stability analysis when linear partial differential equations of evolution are discretized by finite differences.