UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 1.30 to 4.30

## PAPER 7

## HARMONIC ANALYSIS

*Attempt* **THREE** *questions.*

*There are* **five** *questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** What does it mean for a function $u$ on an open subset $\Omega$ of $\mathbf{R}^{d+1}$ to be *harmonic*? State a characterisation of harmonicity in terms of averages.

Show that there exists an isometry $P$ from the Banach space $M(\mathbf{R}^d)$ of bounded Borel measures on $\mathbf{R}^d$ onto $h^1(H^{d+1})$, the Banach space of harmonic functions $u$ on the upper half-space $H^{d+1}$ for which

$$\|u\|_{h^1} = \sup_{t>0}\{\int_{\mathbf{R}^d} |u(x,t)|\, d\lambda(x)\} < \infty\,.$$

Show that there exists a constant $K_d$ such that if $u \in h^1(\mathbf{R}^{d+1})$ then

$$|u(x,t)| \leqslant K_d\, \|u\|_{h^1} / t^d\,.$$

**2** Suppose that $\nu$ is a Borel measure on $\mathbf{R}^d$. Define the *Hardy-Littlewood maximal function* $m(\nu)$ and the related maximal function $m_u(\nu)$, and show that they are equivalent.

Suppose that $\alpha > 0$, and let $E_\alpha = \{x : m_u(\nu)(x) > \alpha\}$. Show that $\lambda(E_\alpha) \leqslant (3^d/\alpha)\nu(E_\alpha)$.

Suppose that $\phi$ is a non-negative strictly decreasing continuous function on $[0,\infty)$ for which $\int_{\mathbf{R}^d} \phi(|x|) d\lambda(x) = 1$.

Show that if $f \in L^1(\mathbf{R}^d)$ then

$$\frac{1}{t^d}\left|\int_{\mathbf{R}^d} \phi\left(\frac{|x-y|}{t}\right) f(y)\, d\lambda(y)\right| \leqslant m(f)(x)\,.$$

Given an example of an application of this result.

**3** Let $(g_n)$ be an $L^1$-bounded martingale on $(0,1]^d$ with respect to the dyadic filtration $(F_n)$ for which $g^* = \sup_n |g_n| \in L^1$. Show that there exists $g \in L^1$ such that $g_n = E(g|F_n)$ for each $n$ and show that $g_n \to g$ in $L^1$-norm and almost everywhere.

Now suppose that$(f_n)$ is an $L^1$-bounded martingale. Let $N > 0$, and let $\tau(x) = \inf\{j : |f_j(x)| \geqslant N\}$. Let $g_n = f_{\tau\wedge n}$ and let $Z(x) = |f_{\tau(x)}|$ if $\tau(x) < \infty$, and let $Z(x) = 0$ otherwise. Show that $(g_n)$ is a martingale, that $g^* \leqslant Z \vee N$ and that $Z \vee N \in L^1$. Use this to show that $f_n$ converges almost everywhere.

[You may assume Doob's Lemma.]

**4** Suppose that $K$ is a function in $L^2(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$ which satisfies

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leqslant C \text{ for all } y \neq 0\,,$$

and which has a bounded Fourier transform. If $f \in L^1 + L^2$, let

$$T(f)(x) = \int K(x-y) f(y)\, dy\,.$$

(i) Show that if $e = \sum_{n=1}^\infty e_n$, where the $(e_n)$ are disjointly supported, then $|T(e)| \leqslant \sum_{n=1}^\infty |T(e_n)|$.

(ii) Show that there exists a constant $L$ such that if $f \in L^1$ is supported in a cube $Q$ and $\int_Q f(x)dx = 0$, and $\hat{Q}$ is the cube with the same centre as $Q$ but with side length $L$ times as big, then

$$\int_{C(\hat{Q})} |T(f)(x)|\, dx \leqslant C\, \|f\|_1\ ,$$

where $C(\hat{Q})$ is the complement of $\hat{Q}$.

(iii) Show that $T$ is of weak type $(1,1)$.

[You may assume Doob's Lemma, and the Martingale Convergence Theorem.]

**5** Suppose that $f$ is an analytic function on $\mathbf{C}^+ = \{z = x + iy \in \mathbf{C} : y > 0\}$ for which

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|\, dx < \infty\,.$$

Let $f^*(x) = \sup\{|f(x'+iy)| : y > 0, |x - x'| \leqslant y\}$. Show that $\int_{-\infty}^\infty f^*(x)\, dx < \infty$.

Explain how this is used to prove the F. and M. Riesz theorem.