UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 6 June, 2005 1.30 to 4.30

PAPER 2

LIE ALGEBRAS AND REPRESENTATION THEORY

Attempt ***QUESTION 1*** *and* ***THREE*** *other questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) Let V be the standard (defining) representation of $\mathfrak{sl}(2)$. Let

$$U = \wedge^2(\mathrm{Sym}^4 V).$$

i) Decompose U into weight spaces, giving a basis for each weight space.

ii) Draw the weight diagram for U, and identify which irreducible representations occur as submodules of U.

iii) Give a basis of each irreducible submodule, and indicate highest weight vectors.

b) Now let V be the standard (defining) representation of $\mathfrak{sl}(3)$, and let V^* be the dual representation. Let

$$W = V \otimes \mathrm{Sym}^2(V^*).$$

i) Draw the weight diagram for W and indicate a basis for each weight space.

ii) Identify the irreducible representations occuring as submodules of W and draw the weight diagram for each irreducible representation.

iii) On their respective weight diagrams indicate a basis for the weight space for each submodule.

2

i) A general element X of $\mathfrak{sp}_4$ is given by

$$X = \begin{bmatrix} a & b & u & v \\ c & d & v & w \\ x & y & -a & -c \\ y & z & -b & -d \end{bmatrix}$$

A general element H of the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sp}_4$ is

$$H = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -s \end{bmatrix}$$

If $L_i : \mathfrak{h} \to \mathbb{C}$ is given by $L_i(H) = H_{ii}$, the ith is diagonal entry of H $(i = 1, \cdots, 4)$ list all the roots of $\mathfrak{sp}_4$ and describe their corresponding root spaces.

ii) On the paper provided, with L_1, L_2 drawn as indicated, draw the weight diagram of

a) the adjoint representation;

b) the defining representation V;

c) $V \otimes V$.

iii) Why might you expect that $V \otimes V$ is not reducible? Draw weight diagrams for two complementary submodules of $V \otimes V$. (N.B. You are not expected to write out a basis or compute weight spaces explicitly.)

3

i) What is meant by the character ring of $\mathbb{Z}[\Lambda_W]$ of a semi-simple Lie algebra $\mathfrak{g}$? If V is a representation of $\mathfrak{g}$, what is char$V$, the character of V? Write down the character of the irreducible representation $\Gamma_{1,2}(=\Gamma_{L_1-2L_3})$.

ii) If V, W are representations of $\mathfrak{g}$, show that

$$\mathrm{char}(V \otimes W) = (\mathrm{char}V)(\mathrm{char}W).$$

iii) Describe the action of the Weyl group on the character ring. Write down Weyl's character formula, explaining any terms or constructions used.

iv) If $u \in \mathbb{Z}[\Lambda_W]$ has the property that

$$W_\alpha u = -u$$

for W_α the element of the Weyl group corresponding to the root α, show that

$$\frac{1}{1-e(\alpha)}u = \left(\sum_{n=0}^{\infty} e\,(n\alpha)\right) u$$

is a well defined element of $\mathbb{Z}[\Lambda_W]$ (i.e., $\sum_{n=0}^{\infty} e(n\alpha)u$ is a finite sum).

4 State and prove Engel's Theorem.

5 Let $\mathfrak{g}$ be a semi-simple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra, with Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \text{root}} \mathfrak{g}_\alpha \,.$$

i) Define the Killing form $B(\ ,\ )$ on $\mathfrak{g}$. Show that if $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, and $H \in \mathfrak{h}$, then

$$B(H, [X, Y]) = \alpha(H) B(X, Y).$$

State clearly any facts about the Killing form you use.

ii) Using the basis of $\mathfrak{g}$

$$\{H_{\alpha_i}, X_\alpha : \alpha \text{ a root}, X_\alpha \in \mathfrak{g}_\alpha,$$
$$\alpha_i \text{ a simple root}, H_{\alpha_i} \text{ the root vector corresponding to } \alpha_i\}$$

or otherwise show that $B(\ ,\ )$ restricted to $\mathbb{R}\{H_\beta\}_{\beta \text{ root}}$, the real span of the root vectors, is real and positive definite.

iii) For $\mathfrak{g} = \mathfrak{sl}_3$, compute $B(E_{12}, E_{21})$.

iv) Define the weight lattice Λ_W and describe how $B(\ ,\ )$ can be used to define a map

$$B : \mathfrak{h} \to \mathfrak{h}^*.$$

Check $B(\mathbb{R}\{H_\beta\}) \subset \mathbb{R}\Lambda_W$.

v) Express $B(H_\alpha)$ as a linear combination of the roots $\{\beta\}$. In the case of $\mathfrak{sl}_3$, express $B(H_{12})$ explicitly ($H_{12} = E_{11} - E_{22}$).

6 Let $\mathfrak{g}$ be a semi-simple Lie algebra. Describe how the Dynkin Diagram corresponding to $\mathfrak{g}$ is obtained. (No proofs need be given, but full marks require a clear exposition of the process and constructions used, indicating where results about semi-simple Lie algebras have been used.)

END OF PAPER