

PAPER 30

CYCLOTOMIC FIELDS

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Notation: Throughout, p will denote an odd prime, and \mathbb{Z}_p the ring of p -adic integers. For any ring A , A^ will denote the multiplicative group of units of A . Let $R = \mathbb{Z}_p[[T]]$ be the ring of formal power series in an indeterminate T with coefficients in \mathbb{Z}_p . Finally, $\phi : R \rightarrow R$ will be the ring homomorphism defined by $\phi(f)(T) = f((1+T)^p - 1)$.*

STATIONERY REQUIREMENTS

*Cover sheet
Treasury Tag
Script paper*

SPECIAL REQUIREMENTS

None

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

- 1 Prove that there exists a unique \mathbb{Z}_p linear map $S : R \rightarrow R$ such that, for all f in R ,

$$(\phi \circ S)(f) = \frac{1}{p} \sum_{\zeta \in \mu_p} f(\zeta(1+T) - 1),$$

where μ_p denotes the group of p -th roots of unity. Prove that S is a left inverse of ϕ , i.e. $S \circ \phi$ is the identity map of R .

- 2 Define the Coleman norm map $N : R^* \rightarrow R^*$. If $f(T)$ in R satisfies

$$\phi(f)(T) \equiv 1 \pmod{p^k R}$$

for some integer $k \geq 1$, prove that $f(T) \equiv 1 \pmod{p^k R}$. Hence show that if f is any element of R^* with $N(f) = f$ and $f \equiv 1 \pmod{pR}$, then necessarily $f = 1$. Let A denote the ring of formal power series in T with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and let W be the subgroup of R^* consisting of all units f such that $Nf = f$. Deduce that A is naturally isomorphic to the direct product of W and A^* .

- 3 Define the Iwasawa algebra $\Lambda(G)$ of any profinite abelian group G , and explain, without detailed proof, its measure-theoretic interpretation.

State Mahler's theorem on continuous functions on \mathbb{Z}_p , and deduce from it a canonical bijection

$$\lambda : R \rightarrow \Lambda(\mathbb{Z}_p).$$

If f is any element of R , prove that, for all integers $k \geq 0$,

$$\int_{\mathbb{Z}_p} x^k d\lambda(f) = (D^k f)(0)$$

where $D = (1+T) \frac{d}{dT}$.

- 4 Write an essay outlining the proof of Iwasawa's theorem, explaining briefly how it led him to formulate the main conjecture on cyclotomic fields.

END OF PAPER