UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Monday 13 June, 2005 1.30 to 4.30

PAPER 38

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) Prove that $X_t = (1-t)\int_0^t \frac{d\beta_s}{1-s}$ satisfies the SDE

$$dX_t = d\beta_t - \tfrac{X_t}{1-t}\,dt, \;\; X(0) = 0, \, t \in [0,1)$$

where β is a one-dimensional Brownian motion. Show that

$$X_t = B_t - (1-t)\int_0^t \tfrac{\beta_s}{(1-s)^2}\,ds$$

b) Prove that if we set $X_1 = 0$, then X_t is the Gaussian process defined on $[0,1]$ with mean $\mathbb{E}(X_t) = 0$ and covariance $\Gamma(s,t) = s(1-t)$, $s \leq t$. (A process X_t, $t \in [0,1]$, is Gaussian if for any family $(t_1, \ldots, t_n)$ in $[0,1]$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is Gaussian.)

c) Show that $\lim_{t\uparrow 1} X_t = 1$ a.s. (Hint: Define $Y_t = X_{1-t}$ and prove that X and Y, as processes, have the same distribution.)

2 a) Two martingales M, $N \in \mathcal{M}_c^2$, $M(0) = N(0) = 0$, are said to be weakly orthogonal if $\mathbb{E}(M_s N_t) = 0$ for all s, $t \geq 0$. Prove that the following are equivalent:

i) M and N are weakly orthogonal,

ii) $\mathbb{E}(M_s N_s) = 0 \;\; \forall\, s \geq 0$,

iii) $\mathbb{E}([M,N]_s) = 0 \;\; \forall\, s \geq 0$ ($[M,N]$ the covariation process of M and N),

iv) $\mathbb{E}(M_T N_s) = 0 \;\; \forall\, s \geq 0$ and stopping time $T \geq s$.

b) Two martingales M, $N \in \mathcal{M}_c^2$, $M(0) = N(0) = 0$, are said to be orthogonal if MN is a martingale. Prove that M, N are orthogonal if and only if $\mathbb{E}(M_T N_s) = 0$ for all $s \geq 0$, T stopping time, $T \leq s$. (You will need to use that X is a martingale if and only if $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for all bounded stopping times T.)

c) Using Kunita Watanabe identity or otherwise, find M, $N \in \mathcal{M}_c^2$, $M(0) = N(0) = 0$ such that M, N are weakly orthogonal but not orthogonal.

d) Prove that if $M \in \mathcal{M}_{c,loc}$ is such that $\mathbb{E}([M]_\infty) < \infty$ then $M \in \mathcal{M}_c^2$. ($[M]$ stands for the quadratic variation process of M.)

3 a) State the existence and uniqueness theorem for maximal local solutions to SDEs with locally Lipschitz coefficients.

b) Let $\sigma : \mathbb{R} \to \mathbb{R}_{>0}$ and $b : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and such that there exists a positive constant K so that

$$\sigma^2(x) + b^2(x) \leq K(1 + x^2)$$

for all $x \in \mathbb{R}$. Let (X, ξ) be the maximal solution to

$$dX_t = \sigma(X_t)d\beta_t + b(X_t)dt, \quad X_0 = 0$$

and $T_n = \inf\{t \geq 0, |X_t| \geq n\}$. Develop $|X_{t\wedge T_n}|^2$ by means of Itô's lemma to prove that $\mathbb{P}(T_n \leq t) \to 0$ as $n \uparrow \infty$ for all $t \geq 0$, and conclude that $\xi = \infty$ a.s..

4 a) Let M be a $(\mathcal{F}_t, \mathbb{P})$ continuous local martingle vanishing at 0 and such that the quadratic variation process $[M]$ is strictly increasing and satisfies $[M] = \infty$ a.s.. Set

$$T_s = \inf\{u, [M]_u \geq s\}.$$

Prove that T_s is a stopping time for each s, and $\beta_t = M_{T_t}$ is a $\mathcal{F}_{T_t}$-Brownian motion such that $M_t = \beta([M]_t)$.

b) Show that if $M \in \mathcal{M}_{c,loc}$ is such that $M(0) = 0$ and $[M]_t$ is deterministic, strictly increasing and such that $[M]_\infty = 0$, then M is a Gaussian martingale and has independent increments. (See question **1**b) for the definition of a Gaussian process)

c) Give an example of a continuous martingale that does not have independent increments.

d) Let β be standard Brownian motion, $\mathcal{F}_t$ its natural filtration. Define $M_t = \beta(t^2)$. Is M a $\mathcal{F}_t$-martingale? If not, find a filtration with respect to which M is a martingale. Find a continuous mapping f and another Brownian motion W such that $M(t) = \int_0^t f(s)dW_s$.

5 Write an essay explaining some connections between the diffusion process X in $\mathbb{R}^d$ with generator

$$Lf(x) = \tfrac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x) \tfrac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x) \tfrac{\partial f}{\partial x^i}$$

and second-order partial differential equations of elliptic and parabolic type. For full credit you should, in particular, establish at least one representation formula for the solution of a partial differential equation in terms of the process X.

6 The following is a stochastic model for two competing species, having populations X_t and Y_t at time t. At an exponential rate of λX_t (respectively $\mu X_t Y_t / N$) the X population increases (respectively decreases) by 1. Independently, at an exponential rate of λY_t (respectively $\mu X_t Y_t / N$) the Y population increases (respectively decreases) by 1. Thus the first change in the total population occurs at an exponential time of rate $\lambda(X_0 + Y_0) + 2\mu X_0 Y_0 / N$. Write down the Lévy kernel for the Markov jump process $(X_t, Y_t)_{t \geq 0}$.

Assume that initially the two populations are equal, of size N. Obtain an approximating differential equation for $(X_t, Y_t)/N$ and comment of the qualitative behaviour of its solution in the two cases $\lambda < \mu$ and $\lambda > \mu$.

Explain how to derive an estimate of the probability that the process $(X_t, Y_t)/N$ deviates by more than a given $\delta > 0$ over a given time interval $[0, t_0]$ from the solution of the differential equation. [*You may assume any form of the exponential martingale inequality.*]

END OF PAPER