UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 8 June, 2005 9 to 11

PAPER 43

ACTUARIAL STATISTICS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In a portfolio of motor insurance policies, the claim sizes $X_1, X_2 \ldots$, are independent identically distributed random variables, independent of the number N of claims in one month. Show that the expected total amount S claimed in one month is $\mathbb{E}N\mathbb{E}(X_1)$, and show that the moment generating function $M_S(t) = \mathbb{E}[e^{St}]$ of S satisfies $M_S(t) = G_N[M_{X_1}(t)]$, where $G_N(z) = \mathbb{E}[z^N]$ is the probability generating function of N and $M_{X_1}(t)$ is the moment generating function of X_1.

Find $\mathbb{E}S$ if X_1 is exponentially distributed with mean μ and $\mathbb{P}(N = k) = q^k p$, $k = 0, 1, 2, \ldots$, where $0 < p = 1 - q < 1$. Show carefully that

$$M_S(t) = \frac{p(1-\mu t)}{p - \mu t} \ ,$$

and hence that the distribution of S has a mass at zero and a density on $(0, \infty)$ given by

$$f_S(x) = \frac{qp}{\mu} e^{-px/\mu}.$$

The insurer takes out excess-of-loss reinsurance with retention $M > 0$. Show carefully that the resulting expected reduction in the insurer's monthly pay-out on this portfolio is

$$\frac{q\mu}{p}\ e^{-M/\mu}.$$

If instead the insurer takes out stop-loss reinsurance, so that the insurer pays $T = \min\{S, \tilde{M}\}$ for some $\tilde{M} > 0$, determine the value of $\tilde{M}$ that makes the expected reduction in monthly pay-out the same as for excess-of-loss.

2 The cumulant generating function $\kappa_X(t)$ of a random variable X is $\kappa_X(t) = \log_e \mathbb{E}(e^{tX})$ and the j^{th} cumulant of X is $\kappa_j = \kappa_X^{(j)}(0)$. Show that κ_1 is the mean μ of X, κ_2 is the variance of X and κ_3 is $\mathbb{E}[(X-\mu)^3]$.

Let $S = X_1 + \ldots + X_N$ where $X_1, X_2, \ldots$ are independent identically distributed positive random variables and N has a Poisson distribution with mean λ, independently of the X_i's. The distribution of S is approximated by that of $V = k + Y$ where Y has density

$$f_Y(y) = \frac{\nu^\alpha y^{\alpha-1} e^{-\nu y}}{\Gamma(\alpha)} \quad , \quad y > 0 \ ,$$

and where k, α and ν are chosen so that the first three cumulants of V match those of S. Determine equations for k, α and ν in terms of λ and $m_j = \mathbb{E}[X_1^j]$, $j = 1, 2, 3$.

Describe the normal approximation to the distribution of S, and discuss the advantages and disadvantages of the normal approximation compared to the approximation above.

3 In a classical risk model, claims arrive in a Poisson process rate λ, the relative safety loading is $\rho > 0$, and the claim sizes $X_1, X_2, \ldots$ are independent, identically distributed positive random variables with mean μ, density $f(x)$ and moment generating function $M(r)$. Assume there exists r_∞, $0 < r_\infty \leqslant \infty$, such that $M(r) \uparrow \infty$ as $r \uparrow r_\infty$. Show that there exists a unique positive solution R to

$$M(r) - 1 = (1+\rho)\mu\, r.$$

By expanding the exponential term in the definition of $M(r)$ as far as the quadratic term, show that an upper bound for R is $r_1 = \frac{2\rho\mu}{\mathbb{E}(X_1^2)}$. Show that there is another upper bound r_2 for R satisfying

$$\mathbb{E}(X_1^3) r_2^2 + 3\mathbb{E}(X_1^2) r_2 - 6\rho\mu = 0.$$

Show that $r_2 < r_1$.

Find R, r_1 and r_2 when $f(x) = \alpha e^{-\alpha x}$, $x > 0$.

4 Write a short paragraph explaining the terms *credibility estimate*, *credibility factor* and *Bayesian credibility.*

The number of claims on a particular risk for n years are $X_1, \ldots, X_n$ where, given θ, $X_1, \ldots, X_n$ are independent with a Poisson distribution with mean θ.

(i) If θ has prior density $\lambda^2 \theta e^{-\lambda\theta}$, find the posterior estimate $\mathbb{E}(X_{n+1}|\mathbf{x})$ of the number of claims in the following year given $\mathbf{X} = (X_1, \ldots, X_n) = \mathbf{x} = (x_1, \ldots, x_n)$, and show that it can be written as a credibility estimate.

(ii) If instead θ has prior density $\pi(\theta)$ and $S_n = X_1 + \ldots + X_n$, show that

$$\mathbb{P}(S_n = s) = \int \frac{e^{-n\theta}(n\theta)^s}{s!} \pi(\theta)\, d\theta, \quad s = 0, 1, \ldots$$

Given a total of s claims in years 1 to n, show that the posterior estimate of X_{n+1} is

$$\frac{(s+1)\mathbb{P}(S_n = s+1)}{n\mathbb{P}(S_n = s)}\,. \tag{$*$}$$

(iii) If $\pi(\theta) = \lambda^2 \theta e^{-\lambda\theta}$, find the distribution of S_n and evaluate $(*)$. Compare your answer to the estimate you obtained in (i).

END OF PAPER