UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 12

PAPER 50

SYMMETRY AND PARTICLE PHYSICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions are of equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Isospin generators $\hat{I}_i$, $i = 1, 2, 3$, satisfy $[\hat{I}_i, \hat{I}_j] = i\epsilon_{ijk}\hat{I}_k$. Using $\hat{I}_\pm = \hat{I}_1 \pm i\hat{I}_2$, show how isospin multiplets may be constructed from a state $|I\,I\rangle$ satisfying $\hat{I}_+|I\,I\rangle = 0$, $\hat{I}_3|I\,I\rangle = I|I\,I\rangle$. Explain why we must have $\hat{I}_-{}^{2I+1}|I\,I\rangle = 0$. If $|I\,M\rangle$ is a normalised state for which $\hat{I}_3$ has the eigenvalue M show that we may define $|I\,M{-}1\rangle$ by

$$\hat{I}_-|I\,M\rangle = \sqrt{(I+M)(I-M+1)}|I\,M{-}1\rangle\,,$$

where

$$\hat{I}_+|I\,M{-}1\rangle = \sqrt{(I+M)(I-M+1)}|I\,M\rangle\,.$$

Calculate $e^{-i\pi\hat{I}_2}\hat{I}_i\,e^{i\pi\hat{I}_2}$ for $i = 1, 2, 3$. Explain why $e^{-i\pi\hat{I}_2}|I\,M\rangle = \alpha_M|I\,{-M}\rangle$ where α_M satisfies $\alpha_{M+1} = -\alpha_M$. Assuming the identity

$$e^{-i\theta\hat{I}_2}|I\,I\rangle = (\cos\tfrac{1}{2}\theta)^{2I}e^{\tan\frac{1}{2}\theta\,\hat{I}_-}|I\,I\rangle\,,$$

show that we must have $\alpha_I = 1$. Hence determine α_M for any M.

The charge conjugation operator $\hat{C}$ satisfies

$$\hat{C}\hat{I}_3\hat{C}^{-1} = -\hat{I}_3\,, \qquad \hat{C}\hat{I}_1\hat{C}^{-1} = -\hat{I}_1\,, \qquad \hat{C}\hat{I}_2\hat{C}^{-1} = \hat{I}_2\,.$$

Show that $\hat{C}\hat{I}_i\hat{C}^{-1}$ obeys the $SU(2)$ Lie algebra and that $\hat{G} = \hat{C}e^{-i\pi\hat{I}_2}$ commutes with $\hat{I}_i$. Assuming for integer I

$$\hat{C}|I\,0\rangle = \eta|I\,0\rangle\,,$$

determine the eigenvalue of $\hat{G}$ for any state $|I\,M\rangle$.

Given that $\pi^0 \to \gamma\gamma$, what is η_{π^0}? Show that for any pion state $\hat{G}|\pi^{\pm,0}\rangle = -|\pi^{\pm,0}\rangle$. The $\rho^{\pm,0}$ and ω are spin one mesons with isospin 1 and 0 respectively and $\eta_{\rho^0} = \eta_\omega = -1$. Explain why we may have the decays under strong interactions $\rho \to \pi\pi$, $\omega \to \pi\pi\pi$ but $\rho \not\to \pi\pi\pi$, $\omega \not\to \pi\pi$.

[*You may use* $[\hat{I}_i, \hat{I}_j\hat{I}_j] = 0$ *and* $\hat{I}_j\hat{I}_j = \hat{I}_-\hat{I}_+ + \hat{I}_3^{\,2} + \hat{I}_3 = \hat{I}_+\hat{I}_- + \hat{I}_3^{\,2} - \hat{I}_3$.]

2 Assuming low mass mesons and baryons have the quark structure $q\bar{q}$ and qqq respectively, where $q = (u, d, s)$ are the three lightest quarks, explain why we may expect the mesons and baryons to belong to representations of $SU(3)$ if the u, d, s masses are nearly equal. What specific $SU(3)$ representations may occur? If I_i are the generators of isospin and the electric charge $Q = I_3 + Y$, plot the baryon states for each possible representation on a diagram with axes I_3, Y.

What is the completely symmetric qqq $SU(3)$ representation? Assuming baryons belonging to this representation have spin $\frac{3}{2}$, why does this suggest that quarks have an additional colour label taking 3 values? Why do we expect that there are no $SU(3)$ singlet baryons at low energies although there are $SU(3)$ singlet mesons?

Explain why a resonance is observed at low energies in $\pi^+ p$ scattering but not in $K^+ p$ scattering?

3 For a Lie algebra $\mathcal{L}$ with a commutator $[X, Y]$, where $[X, [Y, Z]]$ satisfies the Jacobi identity, define the adjoint representation. Assuming $\{T_a\}$ are a basis and that $[T_a, T_b] = c^c{}_{ab} T_c$ show how matrices $T_a^{\rm ad}$ in the adjoint representation satisfying the Lie algebra may be found.

What is meant by the terms simple and semi-simple for Lie algebras? Assuming $\kappa_{ab} = {\rm tr}(T_a^{\rm ad} T_b^{\rm ad})$ has $\det \kappa_{ab} \neq 0$ describe briefly how a semi-simple Lie algebra may be reduced to simple Lie algebras.

For a simple Lie algebra with $\kappa_{ab} = \lambda \delta_{ab}$ show that $[T_a, T_b] = c_{abc} T_c$ with c_{abc} completely antisymmetric.

Let $A_{\mu a}$ be a gauge field and ϕ a field belonging to a representation space for $\mathcal{L}$, on which the generators are t_a. Show that the covariant derivative

$$D_\mu \phi = \partial_\mu \phi + A_{\mu a} t_a \phi$$

satisfies $\delta D_\mu \phi = \lambda_a t_a D_\mu \phi$ with $\delta \phi = \lambda_a t_a \phi$ and with a suitable $\delta A_{\mu a}$, to be specified, for arbitrary infinitesimal $\lambda_a(x)$. Calculate $[D_\mu, D_\nu]\phi = F_{\mu\nu a} t_a \phi$ and hence show that, for the $\delta A_{\mu a}$ just obtained, $\delta F_{\mu\nu a} = c_{abc} \lambda_b F_{\mu\nu c}$. Show that $-\frac{1}{4} F^{\mu\nu}{}_a F_{\mu\nu a}$ is invariant.

4 For a Lie algebra $\mathcal{L}$ describe briefly how a basis $H_i, E_{\pm\boldsymbol{\alpha}}$ may be introduced, where $i = 1, \ldots, r$ and $\{\pm\boldsymbol{\alpha}\}$ are the roots. Show that if $[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}]$ is non zero then $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is a root. What are simple roots $\boldsymbol{\alpha}_i$, $i = 1, \ldots, r$? How is a general root constructed in terms of simple roots? What are positive and negative roots?

For each simple root, let $E_i^{\pm} = E_{\pm\boldsymbol{\alpha}_i}$ and assume $[E_i^+, E_i^-] = \hat{H}_i$ where $[\hat{H}_i, E_i^{\pm}] = \pm 2E_i^{\pm}$. Explain why

$$[E_i^-, E_j^+] = 0\,, \quad j \neq i\,.$$

Assuming standard results for $SU(2)$ show that if $\hat{H}_i|\psi\rangle = -\lambda|\psi\rangle$, $E_i^-|\psi\rangle = 0$ then it is necessary to have $\lambda = 0, 1, 2, \ldots$, and $(E_i^+)^{\lambda+1}|\psi\rangle = 0$. Hence explain why we must have, for $j \neq i$,

$$[\hat{H}_i, E_j^+] = -n_{ij} E_j^+\,, \quad [\underbrace{E_i^+, [\ldots [E_i^+}_{n_{ij}+1}, E_j^+]\ldots]] = 0\,,$$

for some $n_{ij} = 0, 1, 2, \ldots$.

Given a highest weight state $|\mathbf{w}\rangle$ satisfying $\hat{H}_i|\mathbf{w}\rangle = w_i|\mathbf{w}\rangle$, $E_i^+|\mathbf{w}\rangle = 0$ for $w_i = 0, 1, 2 \ldots$, and where $(E_i^-)^{w_i+1}|\mathbf{w}\rangle = 0$, describe in outline how a basis for a representation may be constructed from the action of $E_{\boldsymbol{\alpha}}$ on $|\mathbf{w}\rangle$ for all negative roots $\boldsymbol{\alpha}$.

The Lie algebra for the group $SU(3)$ has $r = 2$ and

$$[\hat{H}_i, E_j^{\pm}] = \pm K_{ji} E_j^{\pm}\,, \qquad (K_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Let $E_3^{\pm} = \pm[E_1^{\pm}, E_2^{\pm}]$. Why must $[E_3^+, E_1^+] = [E_3^+, E_2^+] = 0$? Show that $[E_3^-, E_1^+] = E_2^-$ and $[\hat{H}_1, E_3^{\pm}] = \pm E_3^{\pm}$. Let

$$C = \hat{H}_1^2 + \hat{H}_2^2 + \hat{H}_1\hat{H}_2 + 3(\hat{H}_1 + \hat{H}_2) + 3\sum_{n=1}^{3} E_n^- E_n^+\,.$$

Show that $[C, E_1^+] = [C, \hat{H}_1] = 0$. What is the eigenvalue of C acting on a highest weight state $|w_1, w_2\rangle$?

END OF PAPER