UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 13 June, 2005 1.30 to 4.30

PAPER 68

APPROXIMATION THEORY

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let j_n be the Jackson operator, i.e., for f from $C(\mathbb{T})$, the space of continuous 2π-periodic functions,

$$j_n(f,x) = \int_{-\pi}^{\pi} f(x-t) J_n(t)\,dt, \quad J_n(t) := \tfrac{3}{2\pi n(2n^2+1)} \frac{\sin^4 \frac{nt}{2}}{\sin^4 \frac{t}{2}}\,, \quad \int_{-\pi}^{\pi} J_n(t)\,dt = 1,$$

Prove that, for any $f \in C(\mathbb{T})$, we have the estimate

$$\|j_n(f) - f\| \le c\,\omega_2(f, \tfrac{1}{n})\,,$$

where $\omega_2(f,t)$ is the second modulus of smoothness of f.

2 Let

$$T_n(x) = \cos n \arccos x, \quad x \in [-1,1], \quad n = 0,1,\ldots$$

Prove that T_n satisfies the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

and hence prove that T_n is an algebraic polynomial of degree n. Find its leading coefficient, and the number of its equioscillation points. Finally, *from the first principles* (i.e., without using the Chebyshev alternation theorem), show that $E_{n-1}(f)$, the best approximation to $f(x) = x^n$ from $\mathcal{P}_{n-1}$ on the interval $[-1,1]$, has the value

$$E_{n-1}(f) = 1/2^{n-1}.$$

3 a) Let U be a subspace and f an element of $C(\mathbb{T})$, the space of continuous 2π-periodic functions. Prove that if, for some $p^* \in U$,

$$\operatorname{sign}(f - p^*) \perp U,$$

i.e., if $\int p(x) \operatorname{sign}[f(x) - p^*(x)]\, dx = 0$ for all $p \in U$, then p^* is an element of best approximation to f from U in $L_1(\mathbb{T})$.

b) Prove that, for any $f \in L_1(\mathbb{T})$, and for any $0 < |m| < n$, we have

$$\int_{\mathbb{T}} f(nx) e^{imx} dx = 0,$$

and hence show that, if also $f \perp 1$, then $f(n\cdot)$ is orthogonal to $\mathcal{T}_{n-1}$, the space of trigonometric polynomial of degree $\leq n-1$.

c) Use (a) and (b) to show that, for any α and β, the best approximation to

$$f(x) = \alpha \cos x + \beta \sin x,$$

from $\mathcal{T}_{n-1}$ in $L_1(\mathbb{T})$ is identically zero.

4 Given a knot sequence $\Delta = (t_i)_{i=1}^{n+k}$, let ω_i and $\ell_i(\cdot, t)$ be polynomials in $\mathcal{P}_{k-1}$ defined by

1) $\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1})$,
2) $\ell_i(\cdot, t)$ interpolates $(\cdot - t)_+^{k-1}$ on $x = t_i, ..., t_{i+k-1}$.

Further, let

$$N_i := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the B-spline of order k with the knots $t_i, \ldots, t_{i+k}$.

a) Prove Lee's formula

$$\omega_i(x) N_i(t) = \ell_{i+1}(x, t) - \ell_i(x, t), \quad \forall x, t \in \mathbb{R},$$

and hence derive the Marsden identity:

$$(x - t)^{k-1} = \sum_{i=1}^{n} \omega_i(x) N_i(t), \quad t_k < t < t_{n+1}, \quad \forall x \in \mathbb{R}.$$

b) From the Marsden identity, find the coefficients $a_i^{(m)}$ in the B-spline representation of monomials t^m:

$$t^m = \sum_{i=1}^{n} a_i^{(m)} N_i(t), \quad t_k < t < t_{n+1}, \quad \text{for} \quad m = 0, \ldots, k-1.$$

5 a) Given a knot-sequence $\Delta = (t_i)_{i=1}^{n+k}$, let $(N_i)_{i=1}^{k}$ be the sequence of corresponding B-splines of order k, and let $s_* = \sum_j a_j^* N_j$ be a Chebyshev spline, i.e. a spline such that

$$(-1)^i s_*(x_i^*) = \|s_*\|_\infty = 1,$$

for some increasing sequence $(x_i^*)_{i=1}^{n}$ with $t_i < x_i^* < t_{i+k}$.

Prove that the coefficients a_i^* in the B-spline expansion of such an s_* are given by the formula

$$|a_i^*| = \|\mu_i\|, \qquad i = 1, \ldots, n,$$

where (μ_i) are the functionals dual to (N_j), i.e., $\mu_i(N_j) = \delta_{ij}$.

[*Hint: Prove that the dual functionals* $\mu_i : \mathcal{S}_k(\Delta) \to \mathbb{R}$ *can be defined by the rule*

$$\mu_i(s) = \sum_j A_*^{-1}(i,j) s(x_j), \quad i = 1, \ldots, n,$$

where A_*^{-1} *is the inverse of the collocation matrix* $A_* = (N_j(x_i^*))$.]

b) The B-spline basis of order 3 for the Bernstein knots in $[0,1]$ consists of the quadratic polynomials

$$N_1(x) = x^2, \quad N_2(x) = 2x(1-x), \quad N_3(x) = (1-x)^2.$$

Prove that if

$$p = a_1 N_1 + a_2 N_2 + a_3 N_3, \quad \|p\| \le 1,$$

then

$$|a_1| \le 1, \quad |a_2| \le 3, \quad |a_3| \le 1.$$

6 a) Using the following relation between divided differences

$$[t_0, \ldots, t_k](\cdot - t) f(\cdot) = \gamma_t \, [t_0, \ldots, t_{k-1}] f(\cdot) + (1 - \gamma_t) \, [t_1, \ldots, t_k] f(\cdot) \,, \quad \gamma_t = \tfrac{t - t_0}{t_k - t_0} \,,$$

or otherwise, derive the recurrence formula for B-splines

$$N_{i,k}(t) = \tfrac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \tfrac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t),$$

where $N_{i,m}$ is the B-spline of order m with support $[t_i, t_{i+m}]$ (with L_∞-normalization).

b) Use the B-spline recurrence formula to calculate the values $N_{0,4}(j)$, $j = 1, 2, 3$, for the cubic spline $N_{0,4}$ with integer knots $t_i = i$, $0 \le i \le 4$.

END OF PAPER