UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 13 June, 2005 9 to 12

PAPER 69

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt **THREE** questions from Section A and attempt **ONE** question from Section B.*

Each question from Section B carries twice the weight of a question from Section A.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 The diffusion equation

$$\frac{\partial u}{\partial t} = \nabla^2 u, \qquad x, y \in \mathbf{R}, \quad t \geqslant 0,$$

where $u = u(x,y,t)$, accompanied by the square-integrable initial condition $u(x,y,0) = v(x,y)$, $x, y \in \mathbf{R}$, is solved by the semi-discretized scheme with the stencil

$$\begin{array}{ccc} & \frac{1}{12} & \\ \frac{1}{12} & \frac{2}{3} & \frac{1}{12} \\ & \frac{1}{12} & \end{array} \; u'_{k,l} = \frac{1}{(\Delta x)^2} \begin{array}{ccc} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{2}{3} & -\frac{10}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \; u_{k,l} .$$

a. Determine the value of $p \geqslant 1$ such that the local error of the method is $\mathcal{O}((\Delta x)^{p+1})$.

b. Is the method stable?

2 Consider the two-step ODE method

$$\mathbf{y}_{n+2} - \frac{12}{7}\mathbf{y}_{n+1} + \frac{5}{7}\mathbf{y}_n = h\left[-\frac{2}{7}\mathbf{f}(\mathbf{y}_n) + \frac{4}{7}\mathbf{f}(\mathbf{y}_{n+2})\right].$$

a. What is the order of this method?

b. Is it convergent?

c. Is it A-stable?

3 Let α be a real constant. The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial u}{\partial x}, \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

given with the initial conditions $u(x,0) = \phi(x)$, $0 \leqslant x \leqslant 1$ and zero boundary conditions for $x = 0$ and $x = 1$, is semi-discretized by the method

$$u_m' = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) - \frac{\alpha}{2\Delta x}(u_{m+1} - u_{m-1}).$$

a. Is the method stable?

b. Is the method convergent?

4 Consider the three-stage implicit Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

a. What is the order of the method?

[*Hint: Prove that this is a collocation method.*]

b. Determine the linear stability domain of the method. Is it A-stable?

5 The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

given with initial conditions for $t = 0$ and zero boundary conditions for $x = 0$ and $x = 1$, is solved by the Galerkin method, using hat functions.

a. Derive explicitly the semidiscretized equations.

b. The semidiscretized equations are solved by the forward Euler method. Find the range of the Courant number $\mu = \Delta t/(\Delta x)^2$ for which the method is stable.

SECTION B

6 Write a brief essay on stability analysis of fully-discretized partial differential equations of evolution by using Fourier analysis. You should prove relevant theorems, discuss the advantages and disadvantages of this technique, describe generalization to several space variables and accompany your exposition by examples.

7 Write a brief essay on error and step-size control techniques for discretizations of ordinary differential equations.

END OF PAPER