UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Thursday 9 June, 2005   9 to 12

PAPER 7

BANACH ALGEBRAS

Attempt ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

All Banach algebras should be taken to be over the complex field, and to be non-zero.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let A be a Banach algebra with identity element 1 and let G be the set of all invertible elements of A. Prove that G is an open subset of A and that the mapping $x \mapsto x^{-1}$ $(x \in G)$ is a homeomorphism of G onto itself.

Let (x_n) be a sequence in G and let $x_n \to x$ as $n \to \infty$. Prove that if $x \notin G$ then:

(i) $\|x_n^{-1}\| \to \infty$ as $n \to \infty$;

(ii) the element x has neither left nor right inverse.

Let $a \in A$ and suppose that, for each $\lambda \in \mathbb{C}$, $1 - \lambda a$ has either a left inverse or a right inverse. Prove that $\operatorname{Sp} a = \{0\}$.

2 Let A be a Banach algebra with identity, let $x \in A$ and let U be an open neighbourhood of $\operatorname{Sp} x$ in $\mathbb{C}$. Prove that there is a unique continuous, unital homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(Z) = x$ (where Z is the function $Z(\lambda) = \lambda$ $(\lambda \in U)$).

Prove also that, for every $f \in \mathcal{O}(U)$, $\operatorname{Sp} \Theta_x(f) = f(\operatorname{Sp} x)$.
[*Any form of the Runge approximation theorem may be quoted without proof*.]

Let $x \in A$ have the property that $\operatorname{Sp} x$ contains no real number $t \leqslant 0$. Prove that there is a unique element $y \in A$ such that both $y^3 = x$ and $|\arg \lambda| < \pi/3$ for every $\lambda \in \operatorname{Sp} y$.

3 Let A be a complex Banach algebra with identity, let L be a maximal left ideal of A and let the element a of A be such that $La \subseteq L$. Prove that there is a unique complex number λ such that $a - \lambda 1 \in L$.

Let $Z = \{z \in A : zx = xz \text{ for all } x \in A\}$ (i.e. Z is the *centre* of A). Prove that Z is a closed, commutative subalgebra of A, containing 1. Prove also that if L is any maximal left ideal of A then $L \cap Z$ is a maximal ideal of Z.

4 Let A be a Banach algebra with identity and, for each $x \in A$, let $r(x)$ be the spectral radius of x. Let f be a holomorphic A-valued function on an open subset U of $\mathbb{C}$. Prove that for every compact subset K of U and for every $z \in K$:

(i) $\|f(z)\| \leqslant \sup_{w \in \partial K} \|f(w)\|$;

(ii) $r(f(z)) \leqslant \sup_{w \in \partial K} r(f(w))$.

[*The Dini lemma may be quoted without proof*.]

Let $a, b \in A$ and suppose that, for some constant $C > 0$, $r(a+zb) \leqslant C|z|$ for all $z \in \mathbb{C}\setminus\{0\}$. Prove that $r(a) = 0$.

5 Let A be a Banach algebra with identity. Define what it means for A to have an *involution* $x \mapsto x^*$.

Now suppose that A has an involution. Define what it means for an element x of A to be *hermitian*. Show that the elements 0 and 1 of A are both hermitian. Prove also that, for every $x \in A$:

(i) $\mathrm{Sp}(x^*) = \{\overline{\lambda} : \lambda \in \mathrm{Sp}\, x\}$;

(ii) x is invertible if and only if both xx^* and x^*x are invertible.

Suppose now that A has the additional property that, for every hermitian element h of A, $\mathrm{Sp}\, h \subset \mathbb{R}$. Let B be a closed $*$-subalgebra of A, containing 1 and let $b \in B$. Prove that $\mathrm{Sp}_B(b) = \mathrm{Sp}_A(b)$.

[*Results about the spectrum relative to subalgebras of general Banach algebras may be quoted without proof*.]

END OF PAPER