UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 14 June, 2005 9 to 12

PAPER 71

ASTROPHYSICAL FLUID DYNAMICS

Attempt ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions cary equal weight.

Candidates may bring their notebooks into the examination. The following equations may be assumed.

$$\frac{D\rho}{Dt} + \rho \,\mathrm{div}\, u = 0$$

$$\rho \frac{Du}{Dt} = -\nabla p - \rho \nabla \Phi + j \wedge B$$

$$\rho \frac{De}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} + \ \mathrm{div}\ (\lambda \nabla T) + \epsilon$$

$$\mathrm{div}\, B = 0;\ j = \mu_0^{-1}\,\mathrm{curl}\, B$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\frac{\partial B}{\partial t} = \ \mathrm{curl}\,(u \wedge B)$$

$$p = (\gamma - 1)\rho e = \frac{\mathcal{R}}{\mu}\rho T$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A plane shock wave lies (in the frame of the shock) in the plane $x = 0$. The flow velocity is in the x-direction and of magnitude $U_L\,(U_R)$ to the left (right) of the shock, where left (right) corresponds to the half-space $x < 0\,(x > 0)$. In the same notation, the densities are $\rho_L\,(\rho_R)$, the pressures $p_L\,(p_R)$ and the energy densities $e_L\,(e_R)$. Assuming that the perfect gas law $p = (\gamma - 1)\rho e$ applies on each side of the shock use the Rankine-Hugoniot relations to show that

$$\frac{\rho_R}{\rho_L} = \frac{(\gamma+1)M_L^2}{(\gamma-1)M_L^2+2},$$

where M_L is the Mach number in $x < 0$.

Deduce that $\rho_R > \rho_L \iff M_L^2 > 1$, and hence that the flow must be supersonic on one side of the shock and subsonic on the other.

Show further that

$$\left(\frac{2}{\gamma+1}\right) u_L^2 + u_L\,(u_R - u_L) - \left(\frac{2\gamma}{\gamma+1}\right)\frac{p_L}{\rho_L} = 0\,, \qquad (*)$$

and that

$$\left(\frac{2}{\gamma+1}\right)\frac{\rho_L u_L^2}{p_L} - \frac{p_R}{p_L} - \frac{\gamma-1}{\gamma+1} = 0\,. \qquad (**)$$

Now consider a plane shock lying in the plane $x = X(t) < 0$ and impinging on a stationary solid wall at $x = 0$. Prior to the passage of the shock the gas is at rest with pressure p_0 and density ρ_0. As the shock moves towards the wall with steady velocity $dX/dt = U_+ > 0$, the gas behind the shock has velocity $u_s\,(0 < u_s < U_+)$, pressure p_s and density ρ_s. After the shock has rebounded from the wall it moves with velocity $dx/dt = -U_- < 0$, into the already once-shocked gas. The gas between the shock and the wall is now stationary and has pressure p_1 and density ρ_1. Use (*) to both the pre- and post-rebound configurations to show that $(u_s + U_-)$ and $(u_s - U_+)$ satisfy the same quadratic equation.

Deduce that

$$(u_s - U_+)\,(u_s + U_-) = -\gamma p_s/\rho_s\,. \qquad (\dagger)$$

Similarly apply (**) to both pre- and post-rebound configurations, and hence, using (†) obtain a relationship between p_1/p_s and p_0/p_s, independent of the velocities.

In the case of a strong shock ($p_0 \ll p_s$) show that

$$\frac{p_1}{p_s} = \frac{3\gamma-1}{\gamma-1}.$$

2 Consider a fluid at rest with pressure distribution $p(\mathbf{r})$, density distribution $\rho(\mathbf{r})$ in a fixed gravitational field $\mathbf{g} = -\nabla\Phi(\mathbf{r})$ and permeated by a magnetic field $\mathbf{B}(\mathbf{r})$. The configuration undergoes a small oscillatory perturbation with displacement vector $\boldsymbol{\xi}(\mathbf{r})e^{i\sigma t}$, and with $\operatorname{div}\boldsymbol{\xi} = 0$. If the perturbation to the magnetic field is $\mathbf{b}(\mathbf{r})e^{i\sigma t}$, show that

$$b_i = B_j \frac{\partial \xi_i}{\partial x_j} - \xi_j \frac{\partial B_i}{\partial x_j},$$

and deduce that $\operatorname{div}\mathbf{b} = 0$.

If the density perturbation is $\rho'(\mathbf{r})e^{i\sigma t}$, show that $\rho' = -\boldsymbol{\xi}.\nabla\rho$.

Assuming (without proof) that all surface integrals vanish when integrating by parts show that

$$\begin{aligned}\sigma^2 \int_V \rho \;\xi_i^* \xi_i dV = &\int_V \xi_i^* \xi_j \frac{\partial^2}{\partial x_i \partial x_j}\left[p + \frac{1}{2\mu_0}B^2\right] dV \\ &+ \int_V \rho \;\xi_i^* \xi_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j} dV \\ &+ \frac{1}{\mu_0}\int_V \left(B_j \frac{\partial \xi_i^*}{\partial x_j}\right)\left(B_k \frac{\partial \xi_i}{\partial x_k}\right) dV\end{aligned}$$

and hence that σ^2 is real.

Now consider a particular configuration in which the fluid is vertically stratified $p(z)$, $\rho(z)$ in a constant gravitational field $\mathbf{g} = (0, 0, -g)$, with $g > 0$, and with a horizontal magnetic field $\mathbf{B} = (B(z), 0, 0)$. Write down the equilibrium equation for this configuration.

By considering the perturbation $\boldsymbol{\xi} = (0, 0, \sin ky)$ in the above expression comment on how stability depends on the sign of $\partial\rho/\partial z$.

Comment also on the stability properties of perturbations of the form $\boldsymbol{\xi} = (0, 0, \sin kx)$.

[You may assume that

$$\operatorname{curl}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\operatorname{div}\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{a} + (\mathbf{b}.\nabla)\mathbf{a} - (\mathbf{a}.\nabla)\mathbf{b}\,.]$$

3 The interstellar medium is modelled as a perfect gas subject to cooling per unit volume at the rate $\epsilon(\rho, T) = -\rho^2 \Lambda(T)$, and with thermal conductivity $\lambda(T) = \lambda_0 T^\alpha$, where λ_0 is a constant and $\alpha > 0$. Gravity is neglected. Explain briefly the circumstances for which it is reasonable to assume that the pressure remains uniform, i.e.$\nabla p = 0$.

In this case show that a planar one-dimensional flow obeys the equation

$$\frac{1}{\gamma - 1}\frac{\partial p}{\partial t} + \frac{\gamma}{\gamma - 1} p \frac{\partial v}{\partial x} + \rho^2 \Lambda - \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) = 0\,,$$

where v is the velocity in the x-direction.

Show further that if the flow remains at constant pressure then

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} + \left(\frac{\gamma - 1}{\gamma}\right)\left(\frac{\mu}{\mathcal{R}}\right)^2 p \frac{\Lambda(T)}{T} - \frac{(\gamma - 1)}{\gamma}\frac{\lambda_0 T}{p}\frac{\partial}{\partial x}\left(T^\alpha \frac{\partial T}{\partial x}\right) = 0\,.$$

Using the Lagrangian variable

$$m(x,t) = \int_0^x \rho(x,t)dx,$$

and an appropriately scaled time $\tau = Ct$, where constant C is to be determined, show that this equation can be written in the form

$$\frac{\partial T}{\partial \tau} + \frac{\Lambda(T)}{T} - \lambda_0 \frac{\partial}{\partial m}\left(T^{\alpha-1}\frac{\partial T}{\partial m}\right) = 0\,.$$

At time $t = 0$, gas fills the half space $x > 0$ and has uniform temperature $T = T_0$. The region $x < 0$ contains cold ($T = 0$) infinitely dense gas which does not move but cools infinitely fast. The gas in $x > 0$ cools only by thermal conduction (i.e. $\Lambda = 0$ if $T > 0$). Explain why it is reasonable to seek a similarity solution of the form

$$T(m,\tau) = T_0 f(\xi),$$

with similarity variable $\xi = m/\left(\lambda_0 T_0^{\alpha-1}\tau\right)^{\frac{1}{2}}$, and write down appropriate boundary conditions for $f(\xi)$ at $\xi = 0$ and as $\xi \to \infty$.

If $\lambda(T) = \lambda_0 T$, where λ_0 is a constant, find the function $f(\xi)$ in terms of the function $\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$, and sketch the resulting solution $T(m,\tau)$, indicating the behaviour as τ increases.

Show that the rate L at which energy is radiated by the gas at $x \leqslant 0$ varies as $L \propto t^{-k}$, where k is to be determined.

[Hint: You may assume $\mathrm{erf}\,(\infty) = 1$].

4 An infinite cylinder ($0 \leqslant R \leqslant R_0$) of incompressible fluid with uniform density ρ_0 rotates about the $R = 0$ axis with velocity $\mathbf{u}_0 = (0, R\Omega(R), 0)$, with $\Omega(R) = kR$ where k is a constant.

The fluid is self-gravitating. Show that if the central pressure $p(R = 0) = \pi^2 G^2 \rho_0^3/k^2$ then the radius is $R_0 = (2\pi G\rho_0)^{1/2}/k$, and the effective surface gravity is zero.

The fluid is subject to small perturbations so that the velocity is $\mathbf{u}_0 + \mathbf{u}$, where $\mathbf{u}$ is of the form $\mathbf{u} \propto (u_R(R), u_\phi(R), 0)\exp(i\omega t + im\phi)$.

Show that the perturbation equations are

$$i\sigma u_R - 2\Omega u_\phi = -\frac{\partial W}{\partial R},$$
$$3\Omega u_R + i\sigma u_\phi = -\frac{imW}{R},$$
$$\frac{du_R}{dR} + \frac{u_R}{R} + im\frac{u_\phi}{R} = 0\,,$$

where $\sigma = \omega + m\Omega(R)$ and $W = \dfrac{p'}{\rho} + \Phi'$.

Show that these equations can be reduced to

$$\frac{d^2u_R}{dR^2} + \frac{3}{R}\frac{du_R}{dR} + \frac{u_R}{R^2}\left\{1 - m^2 - \frac{3m\Omega}{\sigma}\right\} = 0\,.$$

Now consider the case $m = 1$. Show that a solution to this equation is

$$u_R = 1 + \frac{kR}{\omega}\,.$$

Assuming that this is the only solution which is regular at $R = 0$, show that the oscillation frequencies obey the equation $\omega^2 = 0$. Give a physical explanation of this result.

[You may assume that in cylindrical polars

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}\,,$$

and that the equations of motion are

$$\frac{\partial v_R}{\partial t} + \mathbf{v}.(\nabla v_R) - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{\partial \Phi}{\partial R}$$
$$\frac{\partial v_\phi}{\partial t} + \mathbf{v}.(\nabla v_\phi) + \frac{v_R v_\phi}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial \phi} - \frac{1}{R}\frac{\partial \Phi}{\partial \phi}\,.]$$

END OF PAPER