UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2011 1:30 pm to 4:30 pm

PAPER 13

ALGEBRAIC TOPOLOGY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be the quotient space $F \times [0,1]/\sim$, where $(x,1) \sim (\phi(x),0)$ for a map $\phi : F \to F$. You may assume that $X \supset F$ has the homotopy type of a cell complex together with a sub-cell-complex. By considering the pairs $(F \times [0,1], F \times \{0,1\})$ and (X,F), or otherwise, construct a long exact sequence

$$\cdots \to H_{i+1}(X) \to H_i(F) \xrightarrow{\phi_* - id_*} H_i(F) \to H_i(X) \to H_{i-1}(F) \xrightarrow{\phi_* - id_*} H_{i-1}(F) \to \cdots$$

Compute $H_*(X;\mathbb{Z})$ in the following two cases:

1. F is the 1-skeleton of a cube, and ϕ is the map which rotates the cube by $2\pi/3$ along a long diagonal;
2. F is the complex projective plane $\mathbb{CP}^2$, and ϕ is the map which in homogeneous co-ordinates takes $[x : y : z] \mapsto [x^2 : y^2 : z^2]$.

Can real projective space $\mathbb{RP}^k$ be obtained from the above construction for some (F,ϕ) ? Justify your answer.

2

A space X_ϕ is obtained from the 3-dimensional torus T^3 by attaching a 2-dimensional closed disc D^2 along a map $\phi : \partial D^2 \to T^3$. Compute the homology groups $H_*(X_\phi;\mathbb{Z})$, explaining carefully how they depend on ϕ.

Prove that T^3 admits two distinct fixed-point-free involutions (homeomorphisms of order two), which are not conjugate in the group of all homeomorphisms. For which homotopy classes of maps ϕ does X_ϕ admit a fixed-point-free involution ? Justify your answer.

Let M be a closed n-dimensional manifold and N be obtained by attaching a single j-disc B^j to M for some $1 \leqslant j \leqslant n$, via a map $S^{j-1} \to M$. If $n = 3$, can N be homotopy equivalent to a closed manifold ? What if $n = 4$? Justify your answers.

3

Define the *cup-product* on cohomology. State the Thom isomorphism theorem, and use it to compute the cohomology ring $H^*(\mathbb{CP}^n;\mathbb{Z})$ of complex projective space.

1. By considering the cup-product on relative cohomology, or otherwise, determine the minimal number of contractible sets needed to cover $\mathbb{CP}^3$.
2. What is the smallest $d > 0$ for which there is a map $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^3$ of degree d ?

Justify your answers.

4

Let $\kappa \subset S^3$ be a *knot* in the 3-dimensional sphere, i.e. the image of a smooth embedding $\phi_\kappa : S^1 \to S^3$ of a circle in the 3-sphere. Compute $H^*(S^3 \backslash \kappa; \mathbb{Z})$.

For κ_1 and κ_2 knots in S^3 with disjoint images, define a *linking number* $lk(\kappa_1, \kappa_2) \in \mathbb{Z}$ with the property that if $lk(\kappa_1, \kappa_2) \neq 0$, then the knots cannot be isotoped into disjoint balls in the 3-sphere. Now view $S^3 = \partial B^4$ as the boundary of the 4-dimensional ball. Suppose κ_1 and κ_2 bound disjoint smoothly embedded surfaces in the 4-ball, each of which has trivial normal bundle. What are the possible values of $lk(\kappa_1, \kappa_2)$? Justify your answer.

The knot κ is *fibred* of genus g if there is a continuous map $\pi : S^3 \backslash \kappa \longrightarrow S^1$ which is a fibre bundle, with fibre the interior of a surface of genus g with boundary κ.

1. Prove that the trivial knot $U = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3$ is fibred of genus 0.
2. If κ is fibred of genus 1, explain how to associate a matrix $A \in SL_2(\mathbb{Z})$ to the fibration. Can the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ arise in this way ? Justify your answer.

5

For closed manifolds M and N of dimension n, define the *connect sum* $M \# N$ of M and N. Prove necessary and sufficient conditions on M and N for $M \# N$ to be orientable.

Let $X = \mathbb{CP}^2 \# \mathbb{CP}^2$. Compute $H^*(X; \mathbb{Z})$ as a ring, giving detailed statements of any general results you invoke.

Let Σ_4 denote a closed surface of genus 4.

1. Prove there is an embedding $\Sigma_4 \hookrightarrow X$ whose image represents a non-trivial homology class. Are any two such embeddings isotopic ?
2. Prove there is *no* embedding $\iota : \Sigma_4 \hookrightarrow X$ for which an open neighbourhood of the image $im(\iota)$ is homeomorphic to the total space of the tangent bundle $T\Sigma_4$.

END OF PAPER