UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2011 1:30 pm to 4:30 pm

PAPER 5

KAC-MOODY AND VIRASORO ALGEBRAS

Attempt no more than ***THREE*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let V be a finite–dimensional real inner product space with orthonormal basis (e_i) and W a finite–dimensional complex inner product space. Let $c : V \to \mathrm{End}(W)$ be a real linear map such that $c(v)^* = c(v)$ and $c(a)c(b) + c(b)c(a) = 2(a, b)I$. If $T \in \mathrm{End}(V)$ satisfies $T^* = -T$ and $\pi(T) = \frac{1}{4}\sum c(Te_i)c(e_i)$ in $\mathrm{End}\, W$, prove that $\pi(T)^* = -\pi(T)$ and $[\pi(T), c(v)] = c(Tv)$ for all $v \in V$.

(b) Prove that if V is a real inner product space of even dimension then any $T \in O(V)$ defines an automorphism of the Clifford algebra of V and that this automorphism is inner, i.e. can be implemented by a unitary in the Clifford algebra.

(c) Prove that the system of bosonic operators a_n, d with $[a_m, a_n] = \delta_{m+n,0}I$, $[d, a_m] = -ma_m$, $a_m^* = a_{-m}$ and $d = d^*$ has a unique irreducible positive energy representation generated by a lowest energy vector v with $a_0 v = \mu v$ and $dv = 0$.

(d) Prove that the operators $L_0 = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n$ and $L_n = \frac{1}{2}\sum_{r+s=n} a_r a_s$ $(n \neq 0)$ satisfy $[L_n, a_m] = -ma_{m+n}$ and give a representation of the Virasoro algebra with central charge $c = 1$.

2

(a) State and prove the double commutant theorem for *–algebras acting on a finite–dimensional complex inner product space.

(b) Let G be a closed subgroup of $GL(V)$ with V a finite–dimensional real vector space. Let $\mathfrak{g} = \{X \in \mathrm{End}(V) : \exp(tX) \in G \text{ for all } t \in \mathbb{R}\}$. Prove that $\mathfrak{g}$ is a real Lie algebra and that the exponential map defines a homeomorphism between a neighbourhood of 0 in $\mathfrak{g}$ and I in G.

(c) Let E, F, H be operators on a finite–dimensional complex inner product space V satisfying $[E, F] = H$, $[H, E] = 2E$, $[H, F] = -2F$, $H = H^*$ and $E^* = F$. Prove that V breaks up as a direct sum of irreducible submodules classified by their dimension $2j + 1$, with j a non–negative half–integer.

(d) Describe *without proof* how to construct a Lie algebra from an even integral lattice in Euclidean space generated by elements with $\|\alpha\|^2 = 2$. State and prove a condition for the Lie algebra to be simple.

3

Either write an essay on how Schur–Weyl duality can be used to prove the Weyl character formula for $U(n)$; **or** write an essay on how the Dirac operator can be used to prove the Weyl character formula for $U(n)$.

4

(a) Explain how to construct bosonic operators using complex fermions. By introducing a shift operator on fermionic Fock space, write down a formula that expresses complex fermions in terms of bosons.

(b) Explain how to construct the affine Kac–Moody algebra $\widehat{\mathfrak{sl}_2}$ at level 1 using complex fermions.

(c) Prove that the lowest energy space of a positive energy irreducible representation of level $\ell \geqslant 1$ must be an irreducible $\mathfrak{sl}_2$ representation of dimension $2j+1$ with $0 \leqslant j \leqslant \ell/2$.

(d) Assuming any results concerning the boson–fermion correspondence that you might require, prove that the level one representations corresponding to $j=0$ and $\frac{1}{2}$ have character $\varphi(q)^{-1}\Theta_{2j,1}(q,z)$, where $\varphi(q)=\prod_{n>0}(1-q^n)$ and $\Theta_{n,m}(q,z)=\sum_{k\in\frac{n}{2m}+\mathbb{Z}} z^{2mk}q^{mk^2}$. Explain briefly how these give rise to representations of the Virasoro algebra with central charge $c=1$.

5

(a) Explain what it means for two fields $A(z)$, $B(w)$ to be *strongly local* and what is meant by their *operator product expansion.* What is meant by a *vertex algebra*? Let the operators e_n be complex fermions satisfying $\{e_m,e_n\}=0$ and $\{e_m,e_n^*\}=\delta_{m,n}I$. Assuming any general properties of strongly local fields that you require, explain how the fields $\psi(z)=\sum e_n z^{-n-1}$ and $\phi(z)=\sum e_n^* z^n$ generate a vertex algebra on fermionic Fock space.

(b) Let $L(z)=\sum L_n z^{-n-2}$ be the Virasoro field in a conformal vertex algebra with central charge c. Prove that

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4}+\frac{2L(w)}{(z-w)^2}+\frac{dL/dw}{z-w}.$$

(c) If $E(z)=\sum E(n)z^{-n-1}$, $F(z)=\sum F(n)z^{-n-1}$ and $H(z)=\sum H(n)z^{-n-1}$ are the fields corresponding to a representation of $\widehat{\mathfrak{sl}_2}$ at level ℓ, show that the vertex algebra they generate has Virasoro field given by the Segal–Sugawara formula

$$L^{\mathfrak{g}}(z)=\frac{1}{2(\ell+2)}\left(\frac{1}{2}:H(z)^2:+:E(z)F(z):+:F(z)E(z):\right).$$

[*You may assume any general properties of vertex algebras that you require.*]

(d) Indicate briefly how the representations in (c) can be used to construct unitary representations of the Virasoro algebra with central charge $c=1-6/m(m+1)$ $(m\geqslant 3)$.

6

Write an essay on the irreducible positive energy representations and the Kac character formula for the affine Kac–Moody algebra $\widehat{\mathfrak{sl}_2}$.

END OF PAPER