UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2011 9:00 am to 12:00 pm

PAPER 64

PHYSICAL COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) From the Friedmann's equation without a cosmological constant:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

where a is the scale factor, G is the gravitational constant, ρ is the matter density, k is the curvature parameter and the other symbols have their usual meaning, derive an expression for the critical density $\rho_{\rm crit}$. Briefly describe the dynamics of a universe with $\rho = \rho_{\rm crit}$ as time $t \to \infty$.

(ii) Consider a universe filled with matter with equation of state

$$p = w\rho c^2 \,.$$

Assume that this matter has density ρ equal to the critical value. Use Friedmann's equation and conservation of energy $(-dU = p\,dV)$ to show that the comoving distance Δr travelled by a photon between redshifts z_1 and z_2 is:

$$\Delta r = \frac{c}{\nu\, H_0}\left\{(1+z_2)^{-\nu} - (1+z_1)^{-\nu}\right\}\,,$$

where H_0 is the value of the Hubble parameter today and

$$\nu = \frac{3w+1}{2}\,.$$

When computing any integrals, assume $w \neq -1/3$.

(iii) If $z_1 \to \infty$ and $z_2 = 0$, your expression gives the comoving particle horizon $\Delta r_{\rm hor}$. Evaluate $\Delta r_{\rm hor}$ for the two cases $w = 0$ and $w = -1$, and briefly explain the answer in the latter case.

(iv) If $z_1 = 0$ and $z_2 = -1$, your expression gives the comoving event horizon $\Delta r'_{\rm hor}$. Evaluate the comoving event horizon $\Delta r'_{\rm hor}$ for the two cases $w = 0$ and $w = -1$, and briefly explain the different behaviour.

(v) For the case $w = -1/3$, show that there are no particle or event horizons. Minkowski space is the metric of empty space, i.e. special relativity. In Minkowski space there are also no particle nor event horizons. Why might a cosmology with the expansion history implied by $w = -1/3$ have the same horizon properties as Minkowski space?

(vi) A photon leaves today $(t = t_0)$ from the present particle horizon. Assuming an Einstein-de Sitter universe, at what time will the photon arrive at the Earth? Express your answer in terms of the present age of the universe.

2

(i) The deceleration parameter is defined by

$$q \equiv -\frac{\ddot{a}}{aH^2}$$

where a is the scale factor and H is the Hubble parameter. Show that in a matter dominated Universe with no cosmological constant

$$q_0 = \frac{\Omega_{\mathrm{m},0}}{2},$$

where the subscript 0 denotes the present time, and $\Omega_{\mathrm{m},0}$ is the ratio of the matter density today to that required for a flat universe.

(ii) Hence show that:

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left[1 - 2q_0 + 2q_0\frac{a_0}{a}\right]$$

(iii) Show that the solutions to the equation at (ii) are as follows:

(A) $q_0 > 1/2$

$$H_0 \cdot t = q_0(2q_0 - 1)^{-3/2}\left[\theta - \sin\theta\right]$$

where the development angle θ is defined by

$$1 - \cos\theta = \left(\frac{2q_0 - 1}{q_0}\right)\frac{a}{a_0}$$

(B) $q_0 = 1/2$

$$H_0 \cdot t = \frac{2}{3}\left(\frac{a}{a_0}\right)^{3/2}$$

(C) $0 < q_0 < 1/2$

$$H_0 \cdot t = q_0(1 - 2q_0)^{-3/2}\left[\sinh\psi - \psi\right]$$

where

$$\cosh\psi - 1 = \left(\frac{1 - 2q_0}{q_0}\right)\frac{a}{a_0}$$

You may use natural units, whereby $c = 1$, in all parts of this question.

3

(i) From the Friedmann's equation without a cosmological constant:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

where a is the scale factor, ρ is the matter density, k is the curvature parameter and the other symbols have their usual meaning show that:

$$\frac{da}{dt} \to \infty \text{ as } t \to 0\,.$$

Then show that the equation:

$$H^2 = H_0^2\left[\Omega_{\rm m,0}(1+z)^3 + \Omega_{\rm k,0}(1+z)^2\right]$$

where H is the Hubble parameter, $\Omega_{\rm m}$ and $\Omega_{\rm k}$ are the contributions of matter and curvature respectively to the critical density, and subscript 0 denotes present time, can be rewritten as:

$$\frac{1}{\Omega_{\rm m}} - 1 = \left(\frac{1}{\Omega_{\rm m,0}} - 1\right)(1+z)^{-1}\,.$$

What does this last equation tell you about the difference between a closed, flat, and open universe at early times?

(ii) Use Friedmann's equation to show that for a uniform comoving population of absorbers, each with constant cross-section, the probability that the line of sight to a distant quasar intersects such an absorber per unit redshift at redshift z is proportional to:

$$\frac{(1+z)^2}{[\Omega_{\rm m,0}(1+z)^3 + \Omega_{\rm k,0}(1+z)^2]^{1/2}}$$

(iii) Does the number of Lyman alpha absorbers per unit redshift evolve in accordance to the above expression between redshifts $z = 2$ and 5? Offer plausible physical explanations for your answer.

(iv) An intergalactic gas cloud produces a Lyman alpha absorption line at redshift $z_{\rm c}$ in the spectrum of a quasar at emission redshift $z_{\rm q}$. Derive an expression between the physical distance $d_{\rm cq}$ of the cloud from the QSO and the redshift difference $\delta z = z_{\rm q} - z_{\rm c}$, under the assumption that δz is small ($\delta z \ll z_{\rm q}$). Give your answer in terms of the parameters $\Omega_{\rm m,0}$ and $\Omega_{\rm k,0}$, assuming $\Omega_{\rm radiation,0} = 0$.

(v) Describe what is meant by the 'proximity effect' in the spectra of distant quasars. Explain how the proximity effect can be used to estimate the intensity of the intergalactic hydrogen ionising background.

(vi) Summarise current ideas on the origin of the ionising background and its evolution with redshift.

4

(i) Give the definition of the distance modulus. The galaxy Cam 300 has a distance modulus of +30; what is its distance in parsecs? Knowing that the Sun has an absolute magnitude in the V-band of $M_{\mathrm{V}} \simeq +5$, estimate the approximate apparent magnitude m_{V} of the galaxy Cam 300, if its stellar mass is similar to that of the Milky Way.

(ii) Starting from the following form of the FRW metric:

$$ds^2 = c^2 dt^2 - a^2(t)\left[dr^2 + f_k(r)^2 (d\theta^2 + \sin^2\theta d\phi^2)\right] ,$$

derive an expression for the present-day ($z = 0$) area A of a sphere formed by photons emitted isotropically from a point source at redshift z_{em}. Assume a $\Omega_{\Lambda,0} = 0$, $\Omega_{\mathrm{m},0} + \Omega_{\mathrm{k},0} = 1$ cosmology. Do *not* compute any integrals over time or redshift explicitly.

(iii) Explain why the luminosity distance to redshift z is given by:

$$d_{\mathrm{L}}(z) = (1+z)\sqrt{\frac{A(z)}{4\pi}} ,$$

where $A(z)$ is your solution to part (ii).

(iv) Compute a Taylor series for $d_{\mathrm{L}}(z)$ about $z = 0$ in terms of H_0 and $\Omega_{\mathrm{m},0}$, evaluating terms of order z^2 but no higher. You can assume that:

$$f_k(r) = \frac{c}{H_0\sqrt{\Omega_{\mathrm{k},0}}} \sinh\left(\frac{H_0\sqrt{\Omega_{\mathrm{k},0}}r}{c}\right) .$$

Give a brief physical explanation of each term in your expansion. Draw a rough sketch of $d_{\mathrm{L}}(z)$ in open, flat and closed universes for small values of z, keeping H_0 fixed.

(v) In a universe without a cosmological constant, would you expect the luminosity distance to redshift $z = 1$ to be greater if $\Omega_{\mathrm{m},0} = 1$ or $\Omega_{\mathrm{m},0} = 0.1$? Give a simple physical explanation for your answer.

(vi) Explain what is meant in astronomy by the term ‘standard candle’. Give two examples of standard candles of particular importance for cosmology, highlighting their main advantages and limitations.

END OF PAPER