UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 1:30 pm to 4:30 pm

PAPER 25

THE RIEMANN ZETA FUNCTION

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define $\zeta(s)$ for $\Re(s) > 1$.

Show that if $\Re(s) > 1$ and $x > 0$ then

$$\zeta(s) = \sum_{n \leqslant x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^\infty \{w\} \frac{dw}{w^{s+1}},$$

where $\{w\} := w - \lfloor w \rfloor$ denotes the fractional part of w.

Deduce that $\zeta(s)$ has a meromorphic continuation to $\Re(s) > 0$, with only a simple pole at $s = 1$.

(b) Prove Van der Corput's Lemma, which states that if $f(x)$ is a real-valued function on an interval $[a, b] \subseteq \mathbb{R}$, and $f'(x)$ is continuous and monotonic on $[a, b]$, and $|f'(x)| \leqslant \delta$ for some $\delta < 1$, then

$$\sum_{a < n \leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} dx + O\left(\frac{1}{1-\delta}\right).$$

[*You may assume basic facts from Fourier analysis and you may assume Abel's summation lemma, provided you state them clearly.*]

Describe *briefly* how one can use Van der Corput's Lemma to prove the Hardy–Littlewood approximation for the zeta function, which states that if $s = \sigma + it$ for some $\sigma > 0$ and $t \in \mathbb{R}$, and if $x \geqslant |t|/\pi$, then

$$\zeta(s) = \sum_{n \leqslant x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}).$$

List a few uses of the Hardy–Littlewood approximation.

2

(a) State and prove the Euler product expression for $\zeta(s)$ when $\Re(s) > 1$.

Prove that if $t \in \mathbb{R}$ and $\sigma \geqslant 1 - \frac{c}{\log^9(|t|+2)}$ then

$$\left|\frac{1}{\zeta(\sigma + it)}\right| = O\left(\log^7(|t| + 2)\right).$$

[*You may assume any standard upper bounds for* $|\zeta(s)|$ *and* $|\zeta'(s)|$*, provided you state them clearly.*]

(b) State von Mangoldt's explicit formula for $\Psi(x)$. Explain *briefly* why the contribution from zeros ρ of $\zeta(s)$ with $\Re(\rho) \leqslant 0$ or $\Re(\rho) \geqslant 1$ may be ignored in the explicit formula. Also state carefully an upper bound for the number of zeros ρ of $\zeta(s)$ satisfying

$$0 < \Re(\rho) < 1 \quad \text{and} \quad t \leqslant \Im(\rho) \leqslant t + 1,$$

where $t \in \mathbb{R}$.

(c) Using parts (a) and (b), or otherwise, prove that if $T \leqslant x$ are large then

$$\Psi(x) = x + O\left(x^{1-c/\log^9 T} \log^2 x\right) + O\left(\frac{x \log^2 x}{T}\right).$$

Deduce the best upper bound you can for $|\Psi(x) - x|$.

Show that if all the zeros ρ of $\zeta(s)$ satisfying $0 < \Re(\rho) < 1$ actually satisfy $\Re(\rho) = 1/2$, then

$$\Psi(x) = x + O\left(x^{1/2} \log^2 x\right).$$

3

Throughout this question you may assume the Euler product expression for the zeta function.

(a) Prove that if $\Re(s) > 1$ then

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Hence prove that if $\sigma > 1$ and $t \in \mathbb{R}$ then

$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - \Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \geqslant 0.$$

(b) State and prove Landau's theorem, which converts upper bounds on $|\zeta(s)|$ into zero-free regions.

[*Provided you state it clearly, you may assume a result that gives a lower bound for* $\Re\frac{f'(z_0)}{f(z_0)}$ *when* f *is a holomorphic function on a disc* $|z - z_0| \leqslant r$, *and* f *is non-zero in the right half of the disc.*]

(c) Suppose that for all large t and all $0 < \sigma \leqslant 1$ we have the Richert bound

$$\zeta(\sigma + it) \ll t^{C(1-\sigma)^{3/2}} \log^{2/3} t,$$

and that for all large t and $\sigma > 1$ we have the bound

$$\zeta(\sigma + it) \ll \log^{2/3} t.$$

Carefully deduce the Vinogradov–Korobov zero-free region for $\zeta(s)$.

[*You may assume standard facts about the zeta function provided you state them clearly, but you may NOT assume any zero-free region results unless you deduce them from part (b) or prove them from scratch.*]

4

(a) Prove that if N is large, and $1 \leqslant M \leqslant N \leqslant t$, and $r := \lfloor \frac{5.01 \log t}{\log N} \rfloor$, then

$$\sum_{N < n \leqslant N+M} n^{-it} = O\left(M \max_{N \leqslant n \leqslant 2N} \frac{|U(n)|}{N^{4/5}} + N^{4/5} + Mt^{-1/500} \right),$$

where

$$U(n) := \sum_{x \leqslant N^{2/5}} \sum_{y \leqslant N^{2/5}} e(\alpha_1 xy + \alpha_2 x^2 y^2 + \ldots + \alpha_r x^r y^r), \qquad \alpha_j := \frac{(-1)^j t}{2\pi j n^j},$$

and $e(z) := e^{2\pi i z}$.

(b) Define Vinogradov's Mean Value $J_{k,r}(Z)$. Prove that if Z is large then

$$J_{k,r}(Z) \geqslant c(k,r) \max\{Z^k, Z^{2k-(1/2)r(r+1)}\},$$

for a suitable constant $c(k,r) > 0$ that may depend on k and r (but not on Z).

Explain *briefly* how one can obtain upper bounds for $|U(n)|$ if one has good upper bounds for $J_{k,r}(N^{2/5})$.

(c) Suppose we knew that for any $1 \leqslant M \leqslant N \leqslant t$,

$$\left| \sum_{N < n \leqslant N+M} n^{-it} \right| \ll M e^{-c(\log^2 N)/\log(t+2)} + N^{4/5}.$$

Deduce that for any large t, and any $0 < \sigma \leqslant 1$, we would have the bound

$$\zeta(\sigma + it) \ll t^{C(1-\sigma)^2} \log^{1/2} t.$$

[*You may assume the Hardy–Littlewood approximation for the zeta function and you may assume Abel's summation lemma, provided you state them clearly.*]

5

(a) State and prove the truncated Perron formula, which relates $\sum_{n \leqslant x} a_n$ with an integral of the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$.

[*You do NOT need to give full details of all the cases of the contour integration argument, but should explain briefly which contours are used and how the integrals are bounded.*]

Deduce that for any large $T \leqslant x$ we have

$$\Psi(x) = \frac{1}{2\pi i} \int_{1+1/\log x - iT}^{1+1/\log x + iT} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) x^s \frac{ds}{s} + O\left(\frac{x \log^2 x}{T} \right).$$

[*You may assume that* $-\zeta'(s)/\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n)/n^s$ *when* $\Re(s) > 1$.]

Deduce further that if $1 \leqslant y \leqslant x$ then

$$\Psi(x+y) - \Psi(x) = O\left(y \int_{-T}^{T} \left| \frac{\zeta'(1+1/\log x + it)}{\zeta(1+1/\log x + it)} \right| dt \right) + O\left(\frac{x \log^2 x}{T} \right).$$

(b) Define the Möbius function $\mu(n)$. Prove that if T and M are large, and $0 < \sigma \leqslant 1$ and $T/2 \leqslant t \leqslant T$, then

$$\zeta(\sigma + it) \sum_{m \leqslant M} \frac{\mu(m)}{m^{\sigma+it}} = 1 + \sum_{\min\{M,T\} < n \leqslant MT} \frac{a_n}{n^{\sigma+it}} + O\left(\frac{M \log M}{T^{\sigma} M^{\sigma}} \right),$$

where $a_n := \sum_{m|n, m \leqslant M, (n/m) \leqslant T} \mu(m)$.

[*You may assume any standard results about the zeta function provided you state them clearly.*]

END OF PAPER