UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS   Part III

Wednesday, 5 June, 2019 1:30 pm to 3:30 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Accretion on to a black hole of mass M is sometimes modelled using Newtonian physics in the gravitational potential

$$\Phi = -\frac{GM}{\left(\sqrt{r^2+z^2} - 2R_\mathrm{g}\right)}\,,$$

where (r, ϕ, z) are cylindrical polar coordinates and $R_\mathrm{g} = GM/c^2$ is the *gravitational radius*.

Calculate the orbital frequency $\Omega(r)$, specific angular momentum $h(r)$, specific energy $\varepsilon(r)$, orbital shear parameter $q(r)$ and epicyclic frequency $\kappa(r)$ of circular orbits of radius $r > 2R_\mathrm{g}$ in this potential. Deduce that these orbits are unstable for $r < r_\mathrm{in}$, where $r_\mathrm{in} = 6R_\mathrm{g}$. Show that the specific energy of a circular orbit of radius r_in is $-\eta c^2$, where $\eta = 1/16$.

(b) The one-dimensional evolutionary equations for an accretion disc are

$$\frac{\partial \mathscr{M}}{\partial t} + \frac{\partial \mathscr{F}}{\partial r} = 0\,,$$
$$\frac{\partial}{\partial t}(\mathscr{M} h) + \frac{\partial}{\partial r}(\mathscr{F} h + \mathscr{G}) = 0\,,$$

where $\mathscr{M}$ is the mass per unit radius, $\mathscr{F}$ is the radial mass flux and $\mathscr{G}$ is the internal torque. Show that the solution for a steady accretion disc with accretion rate $\dot{M}$ and zero torque at $r = r_\mathrm{in}$, in the potential considered in part (a), has vertically integrated viscosity

$$\bar{\nu}\Sigma = \frac{f\dot{M}}{3\pi}\,,$$

where

$$f = \frac{(x-2)}{(x-\frac{2}{3})}\left[1 - \frac{3\sqrt{3}}{\sqrt{2}}\frac{(x-2)}{x\sqrt{x}}\right],$$

with $x = r/R_\mathrm{g}$.

Explain why a zero-torque boundary condition is appropriate at $r = r_\mathrm{in}$, and why the total luminosity of the steady accretion disc can be expected to be $L_\mathrm{disc} = \eta \dot{M} c^2$, where η is the quantity calculated in part (a), if the advection of heat into the black hole can be neglected.

[QUESTION CONTINUES ON THE NEXT PAGE]

(c) The vertical structure of the disc in a steady state is governed by the equations

$$\frac{dp}{dz} = -\rho\Omega^2 z\,,$$

$$\frac{dF_z}{dz} = \rho\nu\left(r\frac{d\Omega}{dr}\right)^2\,,$$

$$F_z = -\frac{16\sigma T^3}{3\kappa\rho}\frac{dT}{dz}\,,$$

$$p = \frac{\mathscr{R}\rho T}{\mu} + \frac{4\sigma T^4}{3c}\,,$$

where the symbols have their usual meanings.

Assume that the opacity, κ, and the ratio of radiation pressure to gas pressure, β, are independent of z. Show that the effective viscosity $\rho\nu$ is equal to

$$\left(\frac{\beta}{1+\beta}\right)\frac{1}{q^2}\frac{c}{\kappa}\,,$$

where q is the orbital shear parameter.

The *Eddington luminosity* L_{E} and the *Eddington accretion rate* $\dot{M}_{\mathrm{E}}$ are defined by $\eta\dot{M}_{\mathrm{E}}c^2 = L_{\mathrm{E}} = 4\pi GMc/\kappa$, where η is the quantity calculated in part (a). Show that the full vertical thickness of the steady accretion disc is

$$\frac{64}{3}\left(\frac{1+\beta}{\beta}\right)q^2 f\frac{\dot{M}}{\dot{M}_{\mathrm{E}}}\frac{GM}{c^2}\,.$$

2

(a) Write down, in terms of cylindrical polar coordinates (r, ϕ, z), the Lagrangian of a particle of unit mass in a gravitational potential $\Phi(r, z)$ that is axisymmetric and has reflectional symmetry in the midplane $z = 0$.

By introducing local coordinates (x, y, z) in the neighbourhood of a reference point that is in a circular orbit of radius r_0 and angular velocity Ω_0 in the midplane, develop the Lagrangian to second order to obtain the expression

$$L_2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + 2\Omega_0 x\dot{y} - \Phi_{\mathrm{t}} \, ,$$

where the tidal potential Φ_{t} is a quadratic function of x and z. Express Φ_{t} in terms of Ω_0 and the partial derivatives of Φ evaluated on the reference orbit.

(b) Derive the equations of motion of the particle in this local model using the quadratic Lagrangian and show that the general solution in the case of a point-mass potential is

$$\begin{aligned} x &= x_0 + \mathrm{Re}\left(A\, e^{-i\Omega_0 t}\right) \, , \\ y &= y_0 - \frac{3}{2}\Omega_0 x_0 t + \mathrm{Re}\left(-2iA\, e^{-i\Omega_0 t}\right) \, , \\ z &= \mathrm{Re}\left(B\, e^{-i\Omega_0 t}\right) \, , \end{aligned}$$

where x_0 and y_0 are real constants and A and B are complex constants.

Interpret the following three quantities that are conserved in this motion:

$$\begin{aligned} p_y &= \frac{1}{2}\Omega_0 x_0 \, , \\ \varepsilon_{\mathrm{h}} &= \frac{1}{2}\Omega_0^2\left(|A|^2 - \frac{3}{4}x_0^2\right) \, , \\ \varepsilon_{\mathrm{v}} &= \frac{1}{2}\Omega_0^2|B|^2 \, . \end{aligned}$$

(c) A dense planetary ring can be modelled, in this local approximation, by a large number of spherical particles of equal size and mass. Neglect gravitational interactions between the particles but assume that they undergo inelastic physical collisions with each other, in which momentum is conserved but kinetic energy is dissipated. Suppose that the particles are placed on circular orbits in the midplane, with initial positions (x_0, y_0) distributed randomly in the region $-a < x_0 < a$. The particles are sufficiently closely packed that collisions occur. Using the results of part (b), or otherwise, explain why the ring spreads symmetrically in the $\pm x$ directions.

3

(a) Consider, within the local model of astrophysical discs, a homogeneous, incompressible disc threaded by a vertical magnetic field, in ideal magnetohydrodynamics. For solutions that are horizontally invariant and have no vertical motion, derive the equations

$$\begin{aligned} \frac{\partial v_x}{\partial t} - 2\Omega v_y &= \frac{B_z}{\mu_0 \rho} \frac{\partial B_x}{\partial z} \,, \\ \frac{\partial v_y}{\partial t} + (2-q)\Omega v_x &= \frac{B_z}{\mu_0 \rho} \frac{\partial B_y}{\partial z} \,, \\ \frac{\partial B_x}{\partial t} &= B_z \frac{\partial v_x}{\partial z} \,, \\ \frac{\partial B_y}{\partial t} + q\Omega B_x &= B_z \frac{\partial v_y}{\partial z} \,, \end{aligned}$$

where $\mathbf{v}$ is the departure from the orbital motion.

(b) Assume that the disc occupies the region $|z| < z^+$, with negligible density outside. Obtain the equilibrium solution corresponding to the boundary conditions $B_x = \pm B_x^+$ and $B_y = 0$ at the upper and lower surfaces $z = \pm z^+$, respectively, where B_x^+ is a constant.

(c) Derive the dispersion relation

$$\left(s^2 + \omega_\mathrm{a}^2\right)\left(s^2 + \omega_\mathrm{a}^2 - 2q\Omega^2\right) + 4\Omega^2 s^2 = 0$$

for perturbations to this equilibrium, where s is the growth rate and ω_a is a quantity to be defined. Deduce that the equilibrium solution obtained in part (b) is unstable if

$$0 < \frac{\pi^2 B_z^2}{8 q \mu_0 \rho z^{+2} \Omega^2} < 1 \,.$$

(d) Show that, if the equilibrium is unstable, then it involves a magnetic field that bends non-monotonically, in the sense that $|B_x|$ does not increase monotonically from the midplane to the surfaces.

[The ideal MHD equations for a homogeneous incompressible fluid in an inertial frame of reference have the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla} \Phi - \frac{1}{\rho} \boldsymbol{\nabla} \Pi + \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B} \,,$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{B} = \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{u} \,,$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \,,$$

where $\Pi = p + \dfrac{|\mathbf{B}|^2}{2\mu_0}$.]

END OF PAPER