UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 1:30 pm to 4:30 pm

PAPER 201

ADVANCED PROBABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(X_n : n \in \mathbb{N})$ be a sequence of independent, identically distributed random variables, with mean 0 and variance 1. Set $S_n = X_1 + \cdots + X_n$. Assume that $|X_1| \leqslant c$ for some constant $c < \infty$. Fix $a, b > 0$ and set

$$T = \inf\{n \geqslant 0 : S_n \leqslant -a \text{ or } S_n \geqslant b\}.$$

(a) Show that $T < \infty$ almost surely, and that $\mathbb{E}(S_T) = 0$ and $\mathbb{E}(S_T^2) = \mathbb{E}(T)$.

(b) Set $p = \mathbb{P}(S_T \geqslant b)$. Show that

$$\frac{a}{a+b+c} \leqslant p \leqslant \frac{a+c}{a+b+c}$$

and deduce that

$$\frac{ab(a+b)}{a+b+c} \leqslant \mathbb{E}(T) \leqslant \frac{(a+c)(b+c)(a+b+2c)}{a+b+c}.$$

[If you use any results of martingale theory then you should state them clearly.]

2 Let $(X_n : n \geqslant 1)$ be a sequence of independent, identically distributed, integrable random variables. Set $S_n = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}(X_1)$.

(a) State and prove the strong law of large numbers.

(b) Show further that, if $X_1 \in L^p(\mathbb{P})$ for some $p > 1$, then

$$\sup_{m \geqslant n} |(S_m/m) - \mu| \to 0$$

in $L^p(\mathbb{P})$ as $n \to \infty$.

[If you use any results of martingale theory then you should state them clearly.]

3

(a) State Prohorov's theorem for a sequence of Borel probability measures on $\mathbb{R}$.

(b) Show that there is a constant $C < \infty$ such that, for all $y \in \mathbb{R}$ and all $\lambda \in (0, \infty)$,

$$1_{\{|y| \geqslant \lambda\}} \leqslant C\lambda \int_0^{1/\lambda} (1 - \cos uy) du.$$

(c) Let $(\mu_n : n \geqslant 1)$ be a sequence of finite Borel measures on $\mathbb{R}$, not necessarily probability measures. For $u \in \mathbb{R}$, set

$$\psi_n(u) = \int_{\mathbb{R}} e^{iuy} \mu_n(dy).$$

Suppose that $\psi_n(u)$ converges for all u as $n \to \infty$, and that the limit $\psi(u)$ is continuous in u at 0. Show that there is a subsequence $(n(k))$ such that $\mu_{n(k)}$ converges weakly on $\mathbb{R}$.

(d) Does the whole sequence $(\mu_n : n \geqslant 1)$ converge weakly? Justify your answer.

[If you use Lévy's continuity theorem then you should prove it.]

4 Let $(X_t)_{0 \leqslant t \leqslant 1}$ be a Brownian motion in $\mathbb{R}$, starting from 0. Define for $0 \leqslant t \leqslant 1$

$$A_t = \int_0^t \frac{X_1 - X_s}{1 - s} ds, \quad M_t = X_t - A_t$$

and denote by $\mathcal{F}_t$ the σ-algebra generated by $(X_s : s \in [0, t] \cup \{1\})$.

(a) Compute $\mathbb{E}(X_t | \mathcal{F}_s)$ for $s, t \in [0, 1]$.

(b) Show that A_1 is almost surely well defined.

(c) Show that $(M_t)_{0 \leqslant t \leqslant 1}$ is an $(\mathcal{F}_t)_{0 \leqslant t \leqslant 1}$-martingale.

(d) Show that $(M_t)_{0 \leqslant t \leqslant 1}$ is a Brownian motion and is independent of X_1.

5 Fix $\varepsilon \in (0, 1/2]$ and define

$$S_0 = \{x \in \mathbb{R}^3 : |x| \leqslant \varepsilon\}, \quad S_1 = \{x \in \mathbb{R}^3 : x + (n, 0, 0) \in S_0 \text{ for some } n \in \mathbb{Z}\}$$

and

$$S_3 = \{x \in \mathbb{R}^3 : x + (n_1, n_2, n_3) \in S_0 \text{ for some } n_1, n_2, n_3 \in \mathbb{Z}\}.$$

Let $(X_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}^3$, starting from x say.

(a) Show that, if $|x| = 1$, then

$$\mathbb{P}_x(X_t \in S_0 \text{ for some } t \geqslant 0) = \varepsilon.$$

(b) Show that, almost surely, $(X_t)_{t \geqslant 0}$ hits S_3 at an unbounded set of times.

(c) Does $(X_t)_{t \geqslant 0}$ hit S_1 at an unbounded set of times? Justify your answer.

6

(a) Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space. What does it mean to say that M is a Poisson random measure on E of intensity μ?

(b) What does it mean to say that $(X_t)_{t \geqslant 0}$ is a Lévy process in $\mathbb{R}$?

(c) Let $(N_t^1)_{t \geqslant 0}, \ldots, (N_t^n)_{t \geqslant 0}$ be independent Poisson processes, of rates $\lambda_1, \ldots, \lambda_n$ respectively, and let $a_1, \ldots, a_n \in \mathbb{R}$. Set

$$X_t = \sum_{k=1}^{n} a_k N_t^k.$$

Show that $(X_t)_{t \geqslant 0}$ is a Lévy process and determine its characteristic exponent ψ.

(d) We call any Lévy process of the form considered in Part (c) a *simple pure-jump Lévy process*. Let K be a Borel measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} y K(dy) < \infty.$$

Set

$$\psi(u) = \int_{(0,\infty)} (e^{iuy} - 1) K(dy).$$

Show that there exists, on some probability space, a sequence of simple pure-jump Lévy processes $(X_t^n)_{t \geqslant 0}$ and a Lévy process $(X_t)_{t \geqslant 0}$ of characteristic exponent ψ such that, as $n \to \infty$,

$$\mathbb{E}\left(\sup_{s \leqslant t} |X_t^n - X_t| \right) \to 0.$$

[Standard properties of integrals with respect to Poisson random measures may be used without proof if stated clearly.]

END OF PAPER