UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 9:00 am to 11:00 am

PAPER 208

CONCENTRATION INEQUALITIES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let X be a real-valued random variable such that $\mathbb{E}X = 0$. Suppose that for some $\nu > 0$,

$$\operatorname{Ent}(e^{\lambda X}) \leqslant \frac{\lambda^2 \nu}{2} \mathbb{E} e^{\lambda X} \quad \text{for } \textit{all } \lambda \in \mathbb{R}.$$

Show that X is sub-Gaussian with variance parameter ν.

(b) Prove a converse of the above statement: If X is a real-valued random variable with $\mathbb{E}X = 0$ and X is sub-Gaussian with variance parameter $\nu/4$, then

$$\operatorname{Ent}(e^{\lambda X}) \leqslant \frac{\lambda^2 \nu}{2} \mathbb{E} e^{\lambda X} \quad \text{for all } \lambda \in \mathbb{R}.$$

[*Hint:* $\operatorname{Ent}(e^{\lambda X})/\mathbb{E}e^{\lambda X} = \mathbb{E}[Z \log Z]$ *for* $Z = e^{\lambda X}/\mathbb{E}e^{\lambda X}$. *Use the concavity of the logarithm and Jensen's inequality.*]

2

(a) Let $Z = f(X_1, \dots, X_n)$, where $X_1, \dots, X_n$ are independent random variables taking values in $\mathcal{X}$ and $f : \mathcal{X}^n \to \mathbb{R}$ is a square-integrable function. State three equivalent forms of the Efron-Stein inequality for bounding $\mathrm{Var}(Z)$.

(b) Consider the setting in part (a). Explain what it means for f to satisfy the bounded-differences property. Derive a bound on $\mathrm{Var}(Z)$ when f satisfies the bounded-differences property.

(c) Let $X_1, \dots, X_n$ be independent random variables supported on the two-point set $\{-1, 1\}$. Let $A \in \mathbb{R}^{n \times m}$ be a fixed $n \times m$ matrix, and define

$$Z = \max_{1 \leqslant \ell \leqslant m} \sum_{k=1}^{n} X_k A_{k,\ell}.$$

Show that

$$\mathrm{Var}(Z) \leqslant \sum_{k=1}^{n} \left(\max_{1 \leqslant \ell \leqslant m} |A_{k,\ell}| \right)^2 .$$

(d) Consider the setting in part (c), with the additional assumption that the random variables $X_1, \dots, X_n$ are i.i.d. and uniformly distributed on $\{-1, 1\}$. Let $X'_1, \dots, X'_n$ be independent copies of $X_1, \dots, X_n$. For $1 \leqslant i \leqslant n$, let

$$Z'_i = \max_{1 \leqslant \ell \leqslant m} \left(\sum_{k \neq i} X_k A_{k,\ell} + X'_i A_{i,\ell} \right) .$$

Show that

$$\mathrm{Var}(Z) \leqslant 2 \max_{1 \leqslant \ell \leqslant m} \left(\sum_{k=1}^{n} A_{k,\ell}^2 \right) .$$

[*Hint: Let* ℓ^* *be the (random) index such that* $Z = \sum_{k=1}^{n} X_k A_{k,\ell^*}$. *Show that* $(Z - Z'_i)_+^2 \leqslant (X_i - X'_i)^2 A_{i,\ell^*}^2$.]

3 Consider $m \geqslant 1$ balls labeled from 1 to m, and $n \geqslant 1$ bins labeled from 1 to n. The balls are thrown independently and uniformly at random into the bins. For $1 \leqslant j \leqslant m$, let X_j be the label of the bin containing ball j. Let $Z = f(X_1, \ldots, X_m)$ be the number of empty bins.

(a) For $1 \leqslant i \leqslant n$, let

$$Z_i = \begin{cases} 1 & \text{if bin } i \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Are the Z_i's independent? Noting that $Z = \sum_{i=1}^n Z_i$, find $\mathbb{E}Z$.

(b) Show that for $t > 0$,

$$\mathbb{P}\left(Z - \mathbb{E}Z \geqslant t\right) \leqslant e^{-2t^2/m} \quad \text{and} \quad \mathbb{P}\left(Z - \mathbb{E}Z \leqslant -t\right) \leqslant e^{-2t^2/m}.$$

(c) For $1 \leqslant j \leqslant m$, define the functions $\alpha_j : \{1, 2, \ldots, n\}^m \to \{0, 1\}$ as

$$\alpha_j(x_1, \ldots, x_m) = \begin{cases} 1 & \text{if ball } j \text{ is the lowest-numbered ball in its bin,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove the following inequality for any $x, y \in \{1, 2, \ldots, n\}^m$:

$$f(y) - f(x) \leqslant \sum_{j=1}^m \alpha_j(x) 1\{x_j \neq y_j\},$$

where $1\{\cdot\}$ is the indicator function.

(d) Using part (c), show that for $t > 0$,

$$\mathbb{P}\left(Z - \mathbb{E}Z \geqslant t\right) \leqslant e^{-t^2/(2n)} \quad \text{and} \quad \mathbb{P}\left(Z - \mathbb{E}Z \leqslant -t\right) \leqslant e^{-t^2/(2n)}.$$

[You may use any result from the lectures without proof provided you state it clearly.]

END OF PAPER