UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2022 9:00 am to 12:00 pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt all **THREE** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Rain falls on a two-dimensional, unconfined aquifer at rainfall rate R, which results in a groundwater table of thickness $h(x,t)$ that feeds a river at $x = 0$ and extends to the drainage divide at $x = L$, across which there is no groundwater flow.

Starting from the equations for flow of water of viscosity μ and density ρ in a porous medium with porosity $1-\phi$ and permeability k derive the leading-order equations for the transient response of the groundwater table in the limit $h \ll L$,

$$(1-\phi)\frac{\partial h}{\partial t} = \frac{k\rho g}{\mu}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + R, \tag{1}$$

where g is the gravitational acceleration.

If the height of the groundwater table at the river $h(0,t) = 0$ derive the response of the groundwater table, providing expressions for the flux into the river $q(0,t)$ and the asymptotic form of the groundwater table near the river ($x \ll L$) in three cases:

(i) the transient groundwater response to rainfall $R > 0$ for $t > 0$ starting from initial condition $h(x,0) = 0$;

(ii) the steady-state groundwater profile when R is constant, as $t \to \infty$;

(iii) the late-time transient drawdown of the aquifer when $R = 0$.

[*You do not need to solve intractable nonlinear equations but should set out clearly what equations need to be solved with what boundary conditions.*]

(b) Now consider the influence of tidal forcing on the flow of the aquifer. Approximate the tidal forcing as a small variation in local gravity,

$$g = g_0(1 + \epsilon e^{i\omega t}),$$

where $\epsilon \ll 1$ and ω is the tidal frequency. The aquifer behaves as an isotropic, poroelastic medium and you may approximate the effective stress as

$$\sigma_e = E\frac{\phi_0 - \phi}{\phi_0},$$

where E is Young's modulus and ϕ_0 is the undeformed solid fraction. Derive an expression for the solid fraction of the aquifer,

$$\phi = \phi_0 \exp[g'(h-z)/l_c],$$

in terms of the variation of gravity $g' = g/g_0$ and the compaction length l_c, which should be expressed in terms of constant physical properties. Show that in the limit $h \ll l_c$, the volume of water per unit horizontal distance is

$$V = (1-\phi_0)h - \phi_0\frac{g'h^2}{2l_c}.$$

Derive an integral expression for the flux of water into the river in terms of V and hence, using scaling or otherwise, estimate the magnitude of the variation in the flux relative to its mean.

2

A solid grows one-dimensionally into a supercooled melt of initial temperature $T_\infty < T_m$, where T_m is the equilibrium freezing temperature of the melt. The rate of solidification is determined kinetically by

$$V = \mathcal{G}(T_m - T_i),$$

where $\mathcal{G}$ is a constant and T_i is the temperature of the solid–melt interface. Show that there is a steady state with $V = \mathcal{G}\Delta T(1 - \mathcal{S})$, provided the Stefan number $\mathcal{S} = L/c_p\Delta T < 1$, where $\Delta T = T_m - T_\infty$, L is the latent heat of fusion and c_p is the specific heat capacity.

Use a linear stability analysis to find a general dispersion relation for the growth rate of perturbations to the interface and thermal field, and show that the interface is marginally stable when

$$\frac{\mathcal{S}}{1-\mathcal{S}} = \Gamma \frac{\lambda}{\lambda - 1}\alpha^2,$$

where

$$\lambda = \frac{1}{2}\left(1 + \sqrt{1 + 4\alpha^2}\right).$$

Here, α is the dimensionless wavenumber, lengths are scaled by κ/V, and $\Gamma = \gamma\mathcal{G}/\kappa$, where γ is the constant of proportionality between the interfacial undercooling related to curvature and the curvature itself, and κ is the thermal diffusivity.

Using sketch graphs and an asymptotic analysis, show that the solid is morphologically unstable if $\mathcal{S} > \Gamma/(1+\Gamma)$ and that the range of unstable wavenumbers becomes infinite as $\mathcal{S} \to 1$.

3

a) A two-dimensional terrestrial ice sheet, treated as a layer of Newtonian fluid of kinematic viscosity ν and thickness $h(x,t)$ flowing over flat bedrock, is subject to a net accumulation rate

$$\begin{aligned} a &= \ \ A \quad \text{if} \quad h > h_s \\ a &= -A \quad \text{if} \quad h < h_s, \end{aligned}$$

where x is horizontal distance, t is time, A is constant, and the elevation of the snow line h_s is constant.

Use thin-film theory to develop evolution equations for $h(x,t)$ in the regions $|x| < x_s$ and $x_s < |x| < x_N$, where $|x| = x_s$ are the horizontal positions of the snow line and $|x| = x_N$ are the termini of the ice sheet. What boundary/interfacial conditions apply to these equations at the ice divide $x = 0$, the termini $x = x_N$ and the snow line $x = x_s$?

Find the steady shape $h(x)$ of the ice sheet and use your solution to determine its maximum thickness h_0, its horizontal extent x_N, and the horizontal position of the snow line

$$x_s = \left(\frac{g}{6\nu A}\right)^{1/2} h_s^2.$$

Deduce that your analysis is appropriate provided that $A \ll g h_s^2/\nu$.

b) Now consider the case of a marine ice sheet in which the bedrock is a constant distance b below sea level. Derive the boundary conditions that apply to the ice sheet at the grounding line $|x| = x_G$. Assuming that for the marine ice sheet $A \ll g' h_G^2/\nu$, determine that the position of the grounding line x_G is given by

$$x_G = 2x_s - \frac{\sqrt{g g'}}{6\nu A} h_G^3,$$

where $g' = g(\rho_w - \rho)/\rho_w$, ρ_w is the density of sea water, and $h_G = b\rho_w/\rho$ is the thickness of the ice sheet at the grounding line. Hence show that the horizontal position of the snow line is now

$$x_s \approx \left(\frac{g}{6\nu A}\right)^{1/2} \left(h_s^4 + \frac{g'}{6\nu A} h_G^6\right)^{1/2}.$$

What is the maximum thickness of the marine ice sheet?

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