UNIVERSITY OF CAMBRIDGE

MAMA/120, NST3AS/120, MAAS/120

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2023 9:00 am to 12:00 pm

PAPER 120

LOGIC AND COMPUTABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Question 1 carries 40 marks. Questions 2 and 3 carry 30 marks each.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What is a *Kripke model*?

(b) State the Completeness Theorem for the Kripke semantics of IPC.

(c) By adequately constructing a Kripke model, show that $(\neg p \to p) \to p$ is not intuitionistically valid.

The formulae of the *implication-free fragment* $IPC \setminus \{\to\}$ *of IPC* are those for which only the connectives $\wedge, \vee, \bot$ and $\top$ appear; its rules are those of IPC, except with the rules for $(\to)$-introduction and $(\to)$-elimination removed and the rule allowing one to conclude $\Gamma \vdash \top$ for any Γ added.

(d) Show that $\phi \wedge (\psi \vee \chi) \vdash_{\mathrm{IPC}\setminus\{\to\}} (\phi \wedge \psi) \vee (\phi \wedge \chi)$ and $(\phi \wedge \psi) \vee (\phi \wedge \chi) \vdash_{\mathrm{IPC}\setminus\{\to\}} \phi \wedge (\psi \vee \chi)$ for all implication-free formulae ϕ, ψ and χ.

[*You may use the Curry-Howard correspondence without proof.*]

(e) Prove that IPC satisfies the disjunction property: if $\vdash_{\mathrm{IPC}} \phi \vee \psi$, then $\vdash_{\mathrm{IPC}} \phi$ or $\vdash_{\mathrm{IPC}} \psi$.

(f) Show that if a proposition is forced by all *finite* Kripke models, then it is intuitionistically valid.

[*In parts (e) and (f) you may assume completeness of the Kripke semantics without proof.*]

2

(a) State and prove the Overspill Lemma for models of Peano arithmetic.

(b) Let $\mathcal{M}$ be a nonstandard model of Peano arithmetic. Is there a formula $\phi(x)$ in the language of arithmetic such that $\mathcal{M} \models \phi(n)$ iff n is a standard natural number? Justify your answer.

(c) Define what it means for a simply typed λ-term M to be in *β-normal form*.

(d) State the Weak Normalisation Theorem for the simply typed λ-calculus.

(e) Does every untyped λ-term admit a reduction to β-normal form? Justify your answer.

(f) Let F be a fixed point combinator in the untyped λ-calculus. Show that there can be no context Γ assigning a type to all the variables in F and a simple type ϕ such that $\Gamma \Vdash F \colon \phi$ in the simply typed λ-calculus.

[*You may assume any results from the lectures that you accurately state without proof.*]

3

(a) Let L be a distributive lattice. What is a *prime filter* of L?

(b) Describe the construction of the *Priestley dual space* $\hat{L}$ of L.

[*You do not need to show any of its properties.*]

(c) State *Stone's Prime Filter-Ideal Lemma.*

(d) Let H be a Heyting algebra and $\hat{H}$ be its Priestley dual space. Show that the identity $(a \Rightarrow b)^* = (\uparrow (a^* \setminus b^*))^{\complement}$ holds for all $a, b \in H$, where $\Rightarrow$ denotes the Heyting implication of H and $(-)^* \colon H \to \mathrm{Clp}\mathcal{D}(\hat{H})$ is the Stone map.

(e) Show that if an implication-free formula ϕ is valid according to (lattice) valuations in any topological space, then it is provable in the implication-free fragment of IPC.

[*You may assume any results that you accurately state from either the lectures or elsewhere in this exam without proof.*]

END OF PAPER