UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Wednesday 5 June 2002 9 to 12

PAPER 3

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***9K, 10K*** *should be in one bundle and* ***1M, 15M*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) Consider the continuous-time Markov chain $(X_t)_{t \geqslant 0}$ on $\{1,2,3,4,5,6,7\}$ with generator matrix

$$Q = \begin{pmatrix} -6 & 2 & 0 & 0 & 0 & 4 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -6 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

Compute the probability, starting from state 3, that X_t hits state 2 eventually.

Deduce that

$$\lim_{t \to \infty} \mathbb{P}(X_t = 2 | X_0 = 3) = \frac{4}{15}.$$

[*Justification of standard arguments is not expected.*]

(ii) A colony of cells contains immature and mature cells. Each immature cell, after an exponential time of parameter 2, becomes a mature cell. Each mature cell, after an exponential time of parameter 3, divides into two immature cells. Suppose we begin with one immature cell and let $n(t)$ denote the expected number of immature cells at time t. Show that

$$n(t) = (4e^t + 3e^{-6t})/7.$$

2K Functional Analysis

(i) Suppose that (f_n) is a decreasing sequence of continuous real-valued functions on a compact metric space (X, d) which converges pointwise to 0. By considering sets of the form $B_n = \{x : f_n(x) < \epsilon\}$, for $\epsilon > 0$, or otherwise, show that f_n converges uniformly to 0.

Can the condition that (f_n) is decreasing be dropped? Can the condition that (X, d) is compact be dropped? Justify your answers.

(ii) Suppose that k is a positive integer. Define polynomials p_n recursively by

$$p_0 = 0, \qquad p_{n+1}(t) = p_n(t) + (t - p_n^k(t))/k.$$

Show that $0 \leqslant p_n(t) \leqslant p_{n+1}(t) \leqslant t^{1/k}$, for $t \in [0, 1]$, and show that $p_n(t)$ converges to $t^{1/k}$ uniformly on $[0, 1]$.

[*You may wish to use the identity* $a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + \ldots + b^{k-1})$.]

Suppose that A is a closed subalgebra of the algebra $C(X)$ of continuous real-valued functions on a compact metric space (X, d), equipped with the uniform norm, and suppose that A has the property that for each $x \in X$ there exists $a \in A$ with $a(x) \neq 0$. Show that there exists $h \in A$ such that $0 < h(x) \leqslant 1$ for all $x \in X$.

Show that $h^{1/k} \in A$ for each positive integer k, and show that A contains the constant functions.

3D Electromagnetism

(i) A plane electromagnetic wave in a vacuum has an electric field

$$\mathbf{E} = (E_1, E_2, 0)\cos(kz - \omega t),$$

referred to cartesian axes (x, y, z). Show that this wave is plane polarized and find the orientation of the plane of polarization. Obtain the corresponding plane polarized magnetic field and calculate the rate at which energy is transported by the wave.

(ii) Suppose instead that

$$\mathbf{E} = (E_1 \cos(kz - \omega t),\ E_2 \cos(kz - \omega t + \phi), 0),$$

with ϕ a constant, $0 < \phi < \pi$. Show that, if the axes are now rotated through an angle ψ so as to obtain an elliptically polarized wave with an electric field

$$\mathbf{E}' = (F_1 \cos(kz - \omega t + \chi),\ F_2 \sin(kz - \omega t + \chi), 0),$$

then

$$\tan 2\psi = \frac{2E_1E_2\cos\phi}{E_1^2 - E_2^2}.$$

Show also that if $E_1 = E_2 = E$ there is an elliptically polarized wave with

$$\mathbf{E}' = \sqrt{2}E\left(\cos(kz - \omega t + \tfrac{1}{2}\phi)\cos\tfrac{1}{2}\phi,\ \sin(kz - \omega t + \tfrac{1}{2}\phi)\sin\tfrac{1}{2}\phi, 0\right).$$

4F Dynamics of Differential Equations

(i) Define the Floquet multiplier and Liapunov exponent for a periodic orbit $\hat{\mathbf{x}}(t)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Show that one multiplier is always unity, and that the other is given by

$$\exp\left(\int_0^T \nabla\cdot \mathbf{f}(\hat{\mathbf{x}}(t))dt\right) \ , \qquad (*)$$

where T is the period of the orbit.

The Van der Pol oscillator $\ddot{x} + \epsilon\dot{x}(x^2-1) + x = 0\,,\ 0 < \epsilon \ll 1$ has a limit cycle $\hat{x}(t) \approx 2\sin t$. Show using $(*)$ that this orbit is stable.

(ii) Show, by considering the normal form for a Hopf bifurcation from a fixed point $\mathbf{x}_0(\mu)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mu)$, that in some neighbourhood of the bifurcation the periodic orbit is stable when it exists in the range of μ for which $\mathbf{x}_0$ is unstable, and unstable in the opposite case.

Now consider the system

$$\left.\begin{aligned} \dot{x} &= x(1-y) + \mu x \\ \dot{y} &= y(x-1) - \mu x \end{aligned}\right\} \qquad x > 0\ .$$

Show that the fixed point $(1+\mu\,,\ 1+\mu)$ has a Hopf bifurcation when $\mu = 0$, and is unstable (stable) when $\mu > 0$ ($\mu < 0$).

Suppose that a periodic orbit exists in $\mu > 0$. Show without solving for the orbit that the result of part (i) shows that such an orbit is unstable. Define a similar result for $\mu < 0$.

What do you conclude about the existence of periodic orbits when $\mu \neq 0$? Check your answer by applying Dulac's criterion to the system, using the weighting $\rho = e^{-(x+y)}$.

5J Representation Theory

Let G be a finite group acting on a finite set X. Define the permutation representation $(\rho, \mathbb{C}[X])$ of G and compute its character π_X. Prove that $\langle \pi_X, 1_G\rangle_G$ equals the number of orbits of G on X. If G acts also on the finite set Y, with character π_Y, show that $\langle \pi_X, \pi_Y\rangle_G$ equals the number of orbits of G on $X \times Y$.

Now let G be the symmetric group S_n acting naturally on the set $X = \{1, \ldots, n\}$, and let X_r be the set of all r-element subsets of X. Let π_r be the permutation character of G on X_r. Prove that

$$\langle \pi_k, \pi_\ell\rangle_G = \ell + 1 \text{ for } 0 \leqslant \ell \leqslant k \leqslant n/2.$$

Deduce that the class functions

$$\chi_r = \pi_r - \pi_{r-1}$$

are irreducible characters of S_n, for $1 \leqslant r \leqslant n/2$.

6J Galois Theory

Show that the polynomial $f(X) = X^5 + 27X + 16$ has no rational roots. Show that the splitting field of f over the finite field $\mathbb{F}_3$ is an extension of degree 4. Hence deduce that f is irreducible over the rationals. Prove that f has precisely two (non-multiple) roots over the finite field $\mathbb{F}_7$. Find the Galois group of f over the rationals.

[*You may assume any general results you need including the fact that A_5 is the only index 2 subgroup of S_5.*]

7J Algebraic Topology

For a finite simplicial complex X, let $b_i(X)$ denote the rank of the finitely generated abelian group H_iX. Define the Euler characteristic $\chi(X)$ by the formula

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Let a_i denote the number of i-simplices in X, for each $i \geqslant 0$. Show that

$$\chi(X) = \sum_i (-1)^i a_i.$$

8K Hilbert Spaces

Let H be an infinite-dimensional, separable Hilbert space. Let T be a compact linear operator on H, and let λ be a non-zero, approximate eigenvalue of T. Prove that λ is an eigenvalue, and that the corresponding eigenspace $E_\lambda(T) = \{x \in H : Tx = \lambda x\}$ is finite-dimensional.

Let S be a compact, self-adjoint operator on H. Prove that there is an orthonormal basis $(e_n)_{n\geqslant 0}$ of H, and a sequence $(\lambda_n)_{n\geqslant 0}$ in $\mathbb{C}$, such that (i) $Se_n = \lambda_n e_n$ $(n \geqslant 0)$ and (ii) $\lambda_n \to 0$ as $n \to \infty$.

Now let S be compact, self-adjoint and *injective*. Let R be a bounded self-adjoint operator on H such that $RS = SR$. Prove that H has an orthonormal basis $(e_n)_{n\geqslant 1}$, where, for every n, e_n is an eigenvector, both of S and of R.

[*You may assume, without proof, results about self-adjoint operators on finite-dimensional spaces.*]

9K Riemann Surfaces

Let α_1, α_2 be two non-zero complex numbers with $\alpha_1/\alpha_2 \notin \mathbb{R}$. Let L be the lattice $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \subset \mathbb{C}$. A meromorphic function f on $\mathbb{C}$ is *elliptic* if $f(z+\lambda) = f(z)$, for all $z \in \mathbb{C}$ and $\lambda \in L$. The *Weierstrass functions* $\wp(z), \zeta(z), \sigma(z)$ are defined by the following properties:

- $\wp(z)$ is elliptic, has double poles at the points of L and no other poles, and $\wp(z) = 1/z^2 + O(z^2)$ near 0;
- $\zeta'(z) = -\wp(z)$, and $\zeta(z) = 1/z + O(z^3)$ near 0;
- $\sigma(z)$ is odd, and $\sigma'(z)/\sigma(z) = \zeta(z)$, and $\sigma(z)/z \to 1$ as $z \to 0$.

Prove the following.
(a) $\wp$, and hence ζ and σ, are uniquely determined by these properties. You are *not* expected to prove the existence of $\wp, \zeta, \sigma$, and you may use Liouville's theorem without proof.
(b) $\zeta(z+\alpha_i) = \zeta(z)+2\eta_i$, and $\sigma(z+\alpha_i) = k_i e^{2\eta_i z}\sigma(z)$, for some constants η_i, k_i $(i = 1, 2)$.
(c) σ is holomorphic, has simple zeroes at the points of L, and has no other zeroes.
(d) Given $a_1, \dots, a_n$ and $b_1, \dots, b_n$ in $\mathbb{C}$ with $a_1 + \dots + a_n = b_1 + \dots + b_n$, the function

$$\frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}$$

is elliptic.
(e) $\wp(u) - \wp(v) = -\dfrac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}$.
(f) Deduce from (e), or otherwise, that $\dfrac{1}{2}\dfrac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)} = \zeta(u+v) - \zeta(u) - \zeta(v)$.

10K Algebraic Curves

Let $f = f(x, y)$ be an irreducible polynomial of degree $n \geq 2$ (over an algebraically closed field of characteristic zero) and $V_0 = \{f = 0\} \subset \mathbb{A}^2$ the corresponding affine plane curve. Assume that V_0 is smooth (non-singular) and that the projectivization $V \subset \mathbb{P}^2$ of V_0 intersects the line at infinity $\mathbb{P}^2 - \mathbb{A}^2$ in n distinct points. Show that V is smooth and determine the divisor of the rational differential $\omega = \dfrac{dx}{f'_y}$ on V. Deduce a formula for the genus of V.

11J Logic, Computation and Set Theory

(i) Explain briefly what is meant by the terms *register machine* and *computable function.*

Let u be the universal computable function $u(m,n) = f_m(n)$ and s a total computable function with $f_{s(m,n)}(k) = f_m(n,k)$. Here $f_m(n)$ and $f_m(n,k)$ are the unary and binary functions computed by the m-th register machine program P_m. Suppose $h : \mathbb{N} \to \mathbb{N}$ is a total computable function. By considering the function

$$g(m,n) = u(h(s(m,m)),n)$$

show that there is a number a such that $f_a = f_{h(a)}$.

(ii) Let P be the set of all partial functions $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Consider the mapping $\Phi : P \to P$ defined by

$$\Phi(g)(m,n) = \begin{cases} n+1 & \text{if } m = 0, \\ g(m-1,1) & \text{if } m > 0,\ n = 0 \text{ and } g(m-1,1) \text{ defined}, \\ g(m-1,g(m,n-1)) & \text{if } mn > 0 \text{ and } g(m-1,g(m,n-1)) \text{ defined}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(a) Show that any fixed point of Φ is a total function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Deduce that Φ has a unique fixed point.
[*The Bourbaki-Witt Theorem may be assumed if stated precisely.*]

(b) It follows from standard closure properties of the computable functions that there is a computable function ψ such that

$$\psi(p,m,n) = \Phi(f_p)(m,n).$$

Assuming this, show that there is a total computable function h such that

$$\Phi(f_p) = f_{h(p)} \text{ for all } p.$$

Deduce that the fixed point of Φ is computable.

12L Probability and Measure

Derive the characteristic function of a real-valued random variable which is normally distributed with mean μ and variance σ^2. What does it mean to say that an $\mathbb{R}^n$-valued random variable has a *multivariate Gaussian distribution*? Prove that the distribution of such a random variable is determined by its ($\mathbb{R}^n$-valued) mean and its covariance matrix.

Let X and Y be random variables defined on the same probability space such that (X,Y) has a Gaussian distribution. Show that X and Y are independent if and only if $\operatorname{cov}(X,Y) = 0$. Show that, even if they are not independent, one may always write $X = aY + Z$ for some constant a and some random variable Z independent of Y.

[*The inversion theorem for characteristic functions and standard results about independence may be assumed.*]

13L Applied Probability

(a) Define a renewal process and a discrete renewal process.

(b) State and prove the Discrete Renewal Theorem.

(c) The sequence $\mathbf{u} = \{u_n : n \geqslant 0\}$ satisfies

$$u_0 = 1, \quad u_n = \sum_{i=1}^{n} f_i u_{n-i}, \qquad \text{for } n \geqslant 1$$

for some collection of non-negative numbers $(f_i : i \in \mathbb{N})$ summing to 1. Let $U(s) = \sum_{n=1}^{\infty} u_n s^n$, $F(s) = \sum_{n=1}^{\infty} f_n s^n$. Show that

$$F(s) = \frac{U(s)}{1+U(s)}.$$

Give a probabilistic interpretation of the numbers u_n, f_n and $m_n = \sum_{i=1}^{n} u_i$.

(d) Let the sequence u_n be given by

$$u_{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}, \quad u_{2n+1} = 0, \quad n \geqslant 1.$$

How is this related to the simple symmetric random walk on the integers $\mathbb{Z}$ starting from the origin, and its subsequent returns to the origin? Determine $F(s)$ in this case, either by calculating $U(s)$ or by showing that F satisfies the quadratic equation

$$F^2 - 2F + s^2 = 0, \qquad \text{for} \quad 0 \leqslant s < 1.$$

14L Optimization and Control

Consider a scalar system with $x_{t+1} = (x_t + u_t)\xi_t$, where $\xi_0, \xi_1, \ldots$ is a sequence of independent random variables, uniform on the interval $[-a, a]$, with $a \leqslant 1$. We wish to choose $u_0, \ldots, u_{h-1}$ to minimize the expected value of

$$\sum_{t=0}^{h-1}(c + x_t^2 + u_t^2) + 3x_h^2\,,$$

where u_t is chosen knowing x_t but not ξ_t. Prove that the minimal expected cost can be written $V_h(x_0) = hc + x_0^2\Pi_h$ and derive a recurrence for calculating $\Pi_1, \ldots, \Pi_h$.

How does your answer change if u_t is constrained to lie in the set $\mathcal{U}(x_t) = \{u : |u + x_t| < |x_t|\}$?

Consider a stopping problem for which there are two options in state x_t, $t \geqslant 0$:

(1) stop: paying a terminal cost $3x_t^2$; no further costs are incurred;

(2) continue: choosing $u_t \in \mathcal{U}(x_t)$, paying $c + u_t^2 + x_t^2$, and moving to state $x_{t+1} = (x_t + u_t)\xi_t$.

Consider the problem of minimizing total expected cost subject to the constraint that no more than h continuation steps are allowed. Suppose $a = 1$. Show that an optimal policy stops if and only if either h continuation steps have already been taken or $x^2 \leqslant 2c/3$.

[*Hint: Use induction on h to show that a one-step-look-ahead rule is optimal. You should not need to find the optimal u_t for the continuation steps.*]

15M Principles of Statistics

(i) Describe in detail how to perform the Wald, score and likelihood ratio tests of a *simple* null hypothesis $H_0 : \theta = \theta_0$ given a random sample $X_1, \ldots, X_n$ from a regular one-parameter density $f(x; \theta)$. In each case you should specify the asymptotic null distribution of the test statistic.

(ii) Let $X_1, \ldots, X_n$ be an independent, identically distributed sample from a distribution F, and let $\hat{\theta}(X_1, \ldots, X_n)$ be an estimator of a parameter θ of F.

Explain what is meant by: (a) the *empirical distribution function* of the sample; (b) the *bootstrap estimator* of the *bias* of $\hat{\theta}$, based on the empirical distribution function. Explain how a bootstrap estimator of the *distribution function* of $\hat{\theta} - \theta$ may be used to construct an approximate $1 - \alpha$ confidence interval for θ.

Suppose the parameter of interest is $\theta = \mu^2$, where μ is the mean of F, and the estimator is $\hat{\theta} = \bar{X}^2$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ is the sample mean.

Derive an *explicit* expression for the bootstrap estimator of the bias of $\hat{\theta}$ and show that it is biased as an estimator of the true bias of $\hat{\theta}$.

Let $\hat{\theta}_i$ be the value of the estimator $\hat{\theta}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ computed from the sample of size $n-1$ obtained by deleting X_i and let $\hat{\theta}_{.} = n^{-1} \sum_{i=1}^{n} \hat{\theta}_i$. The *jackknife* estimator of the bias of $\hat{\theta}$ is

$$b_J = (n-1)\ (\hat{\theta}_{.} - \hat{\theta})\ .$$

Derive the jackknife estimator b_J for the case $\hat{\theta} = \bar{X}^2$, and show that, as an estimator of the true bias of $\hat{\theta}$, it is unbiased.

16L Stochastic Financial Models

(i) Explain briefly what it means to say that a stochastic process $\{W_t,\ t \geqslant 0\}$ is a standard Brownian motion.

Let $\{W_t,\ t \geqslant 0\}$ be a standard Brownian motion and let a, b be real numbers. What condition must a and b satisfy to ensure that the process e^{aW_t+bt} is a martingale? Justify your answer carefully.

(ii) At the beginning of each of the years $r = 0, 1, \ldots, n-1$ an investor has income X_r, of which he invests a proportion ρ_r, $0 \leqslant \rho_r \leqslant 1$, and consumes the rest during the year. His income at the beginning of the next year is

$$X_{r+1} = X_r + \rho_r X_r W_r,$$

where $W_0, \ldots, W_{n-1}$ are independent positive random variables with finite means and $X_0 \geqslant 0$ is a constant. He decides on ρ_r after he has observed both X_r and W_r at the beginning of year r, but at that time he does not have any knowledge of the value of W_s, for any $s > r$. The investor retires in year n and consumes his entire income during that year. He wishes to determine the investment policy that maximizes his expected total consumption

$$\mathbb{E}\left[\sum_{r=0}^{n-1}(1-\rho_r)X_r + X_n\right].$$

Prove that the optimal policy may be expressed in terms of the numbers $b_0, b_1, \ldots, b_n$ where $b_n = 1$, $b_r = b_{r+1} + \mathbb{E}\max(b_{r+1}W_r, 1)$, for $r < n$, and determine the optimal expected total consumption.

17F Dynamical Systems

Let $\mathcal{A}$ be a finite alphabet of letters and Σ either the semi-infinite space or the doubly infinite space of sequences whose elements are drawn from $\mathcal{A}$. Define the natural topology on Σ. If W is a set of words, denote by Σ_W the subspace of Σ consisting of those sequences none of whose subsequences is in W. Prove that Σ_W is a closed subspace of Σ; and state and prove a necessary and sufficient condition for a closed subspace of Σ to have the form Σ_W for some W.

If $\mathcal{A} = \{0, 1\}$ and

$$W = \{000,\ 111,\ 010,\ 101\}$$

what is the space Σ_W?

18G Partial Differential Equations

Define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. State the Fourier inversion theorem for the Fourier transform of a Schwartz function.

Consider the initial value problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u = 0\ ,\ x \in \mathbb{R}^n\ ,\ 0 < t < \infty\ ,$$

$$u(0,x) = f(x)\ , \qquad \frac{\partial u}{\partial t}(0,x) = 0$$

for f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Show that the solution can be written as

$$u(t,x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(t,\xi) d\xi\ ,$$

where

$$\hat{u}(t,\xi) = \cos\left(t\sqrt{1+|\xi|^2}\right) \hat{f}(\xi)$$

and

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

State the Plancherel-Parseval theorem and hence deduce that

$$\int_{\mathbb{R}^n} |u(t,x)|^2 dx \le \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

19G Methods of Mathematical Physics

Show that the equation

$$zw'' + w' + (\lambda - z)w = 0$$

has solutions of the form

$$w(z) = \int_\gamma (t-1)^{(\lambda-1)/2}(t+1)^{-(\lambda+1)/2} e^{zt} dt.$$

Give examples of possible choices of γ to provide two independent solutions, assuming $\mathrm{Re}(z) > 0$. Distinguish between the cases $\mathrm{Re}\,\lambda > -1$ and $\mathrm{Re}\,\lambda < 1$. Comment on the case $-1 < \mathrm{Re}\ \lambda < 1$ and on the case that λ is an odd integer.

Show that, if $\mathrm{Re}\,\lambda < 1$, there is a solution $w_1(z)$ that is bounded as $z \to +\infty$, and that, in this limit,

$$w_1(z) \sim A\, e^{-z} z^{(\lambda-1)/2} \left(1 - \frac{(1-\lambda)^2}{8z}\right) ,$$

where A is a constant.

20F Numerical Analysis

(i) Determine the order of the multistep method

$$\mathbf{y}_{n+2} - (1+\alpha)\mathbf{y}_{n+1} + \alpha\mathbf{y}_n = h[\tfrac{1}{12}(5+\alpha)\mathbf{f}_{n+2} + \tfrac{2}{3}(1-\alpha)\mathbf{f}_{n+1} - \tfrac{1}{12}(1+5\alpha)\mathbf{f}_n]$$

for the solution of ordinary differential equations for different choices of α in the range $-1 \leqslant \alpha \leqslant 1$.

(ii) Prove that no such choice of α results in a method whose linear stability domain includes the interval $(-\infty,\ 0)$.

21E Foundations of Quantum Mechanics

(i) Two particles with angular momenta j_1, j_2 and basis states $|j_1\ m_1\rangle, |j_2\ m_2\rangle$ are combined to give total angular momentum j and basis states $|j\ m\rangle$. State the possible values of j, m and show how a state with $j = m = j_1 + j_2$ can be constructed. Briefly describe, for a general allowed value of j, what the Clebsch-Gordan coefficients are.

(ii) If the angular momenta j_1 and j_2 are both 1 show that the combined state $|2\ 0\rangle$ is

$$|2\ 0\rangle = \sqrt{\frac{1}{6}}\Big(|1\ 1\rangle|1\ {-1}\rangle + |1\ {-1}\rangle|1\ 1\rangle\Big) + \sqrt{\frac{2}{3}}|1\ 0\rangle|1\ 0\rangle.$$

Determine the corresponding expressions for the combined states $|1\ 0\rangle$ and $|0\ 0\rangle$, assuming that they are respectively antisymmetric and symmetric under interchange of the two particles.

If the combined system is in state $|0\ 0\rangle$ what is the probability that measurements of the z-component of angular momentum for either constituent particle will give the value of 1?

[*Hint:* $J_\pm|j\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m \pm 1\rangle$.]

22D Statistical Physics

A system consisting of non-interacting bosons has single-particle levels uniquely labelled by r with energies ϵ_r, $\epsilon_r \geq 0$. Show that the free energy in the grand canonical ensemble is

$$F = kT \sum_r \log(1 - e^{-\beta(\epsilon_r - \mu)}) .$$

What is the maximum value for μ?

A system of N bosons in a large volume V has one energy level of energy zero and a large number $M \gg 1$ of energy levels of the same energy ϵ, where M takes the form $M = AV$ with A a positive constant. What are the dimensions of A?

Show that the free energy is

$$F = kT \left(\log(1 - e^{\beta\mu}) + AV \log(1 - e^{-\beta(\epsilon - \mu)}) \right) .$$

The numbers of particles with energies $0, \epsilon$ are respectively N_0, N_ϵ. Write down expressions for N_0, N_ϵ in terms of μ.

At temperature T what is the maximum number of bosons N_ϵ^{max} in the normal phase (the state with energy ϵ)? Explain what happens when $N > N_\epsilon^{max}$.

Given N and T calculate the transition temperature T_B at which Bose condensation occurs.

For $T > T_B$ show that $\mu = \epsilon(T_B - T)/T_B$. What is the value of μ for $T < T_B$?

Calculate the mean energy E for (a) $T > T_B$ (b) $T < T_B$, and show that the heat capacity of the system at constant volume is

$$C_V = \begin{cases} \dfrac{1}{kT^2} \dfrac{AV\epsilon^2}{(e^{\beta\epsilon} - 1)^2} & T < T_B \\ 0 & T > T_B . \end{cases}$$

23E Applications of Quantum Mechanics

A periodic potential is expressed as $V(\mathbf{x}) = \sum_{\mathbf{g}} a_{\mathbf{g}}\, e^{i\mathbf{g}\cdot\mathbf{x}}$, where $\{\mathbf{g}\}$ are reciprocal lattice vectors and $a_{\mathbf{g}}{}^* = a_{-\mathbf{g}}$, $a_{\mathbf{0}} = 0$. In the nearly free electron model explain why it is appropriate, near the boundaries of energy bands, to consider a Bloch wave state

$$|\psi_{\mathbf{k}}\rangle = \sum_r \alpha_r\, |\mathbf{k}_r\rangle\,, \qquad \mathbf{k}_r = \mathbf{k} + \mathbf{g}_r\,,$$

where $|\mathbf{k}\rangle$ is a free electron state for wave vector $\mathbf{k}$, $\langle\mathbf{k}'|\mathbf{k}\rangle = \delta_{\mathbf{k}'\mathbf{k}}$, and the sum is restricted to reciprocal lattice vectors $\mathbf{g}_r$ such that $|\mathbf{k}_r| \approx |\mathbf{k}|$. Obtain a determinantal formula for the possible energies $E(\mathbf{k})$ corresponding to Bloch wave states of this form.

[*You may take* $\mathbf{g}_1 = \mathbf{0}$ *and assume* $e^{i\mathbf{b}\cdot\mathbf{x}}|\mathbf{k}\rangle = |\mathbf{k}+\mathbf{b}\rangle$ *for any* $\mathbf{b}$.]

Suppose the sum is restricted to just $\mathbf{k}$ and $\mathbf{k}+\mathbf{g}$. Show that there is a gap $2|a_{\mathbf{g}}|$ between energy bands. Setting $\mathbf{k} = -\frac{1}{2}\mathbf{g} + \mathbf{q}$, show that there are two Bloch wave states with energies near the boundaries of the energy bands

$$E_{\pm}(\mathbf{k}) \approx \frac{\hbar^2|\mathbf{g}|^2}{8m} \pm |a_{\mathbf{g}}| + \frac{\hbar^2|\mathbf{q}|^2}{2m} \pm \frac{\hbar^4}{8m^2|a_{\mathbf{g}}|}\,(\mathbf{q}\cdot\mathbf{g})^2\,.$$

What is meant by effective mass? Determine the value of the effective mass at the top and the bottom of the adjacent energy bands if $\mathbf{q}$ is parallel to $\mathbf{g}$.

24C Fluid Dynamics II

(i) Suppose that, with spherical polar coordinates, the Stokes streamfunction

$$\Psi_\lambda(r,\theta) = r^\lambda \sin^2\theta\cos\theta$$

represents a Stokes flow and thus satisfies the equation $D^2(D^2\Psi_\lambda) = 0$, where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\ .$$

Show that the possible values of λ are 5, 3, 0 and -2. For which of these values is the corresponding flow irrotational? Sketch the streamlines of the flow for the case $\lambda = 3$.

(ii) A spherical drop of liquid of viscosity μ_1, radius a and centre at $r = 0$, is suspended in another liquid of viscosity μ_2 which flows with streamfunction

$$\Psi \sim \Psi_\infty(r,\theta) = \alpha r^3 \sin^2\theta\cos\theta$$

far from the drop. The two liquids are of equal densities, surface tension is sufficiently strong to keep the drop spherical, and inertia is negligible. Show that

$$\Psi = \begin{cases} (Ar^5 + Br^3)\sin^2\theta\cos\theta & (r < a), \\ (\alpha r^3 + C + D/r^2)\sin^2\theta\cos\theta & (r > a) \end{cases}$$

and obtain four equations determining the constants A, B, C and D. (You need not solve these equations.)

[*You may assume, with standard notation, that*

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\quad,\quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}\quad,\quad e_{r\theta} = \frac{1}{2}\left\{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right\}.]$$

25C Waves in Fluid and Solid Media

Consider the equation

$$\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} - \frac{\partial^3\phi}{\partial x^3} = 0.$$

Find the dispersion relation for waves of frequency ω and wavenumber k. Do the wave crests move faster or slower than a packet of waves?

Write down the solution with initial value

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk\ ,$$

where $A(k)$ is real and $A(-k) = A(k)$.

Use the method of stationary phase to obtain an approximation to $\phi(x,t)$ for large t, with x/t having the constant value V. Explain, using the notion of group velocity, the constraint that must be placed on V.