UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 4 June, 2012 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *…,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Show that the continued fraction for $\sqrt{13}$ is $[3; \overline{1,1,1,1,6}]$.

Hence, or otherwise, find a solution to the equation $x^2 - 13y^2 = 1$ in positive integers x and y. Write down an expression for another solution.

2F Topics in Analysis

State a version of the Baire category theorem for a complete metric space. Let T be the set of real numbers x with the property that, for each positive integer n, there exist integers p and q with $q \geqslant 2$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}\,.$$

Is T an open subset of $\mathbb{R}$? Is T a dense subset of $\mathbb{R}$? Justify your answers.

3G Geometry and Groups

Let G be a crystallographic group of the Euclidean plane. Define the *lattice* and the *point group* of G. Suppose that the lattice for G is $\{(k,0) : k \in \mathbb{Z}\}$. Show that there are five different possibilities for the point group. Show that at least one of these point groups can arise from two groups G that are not conjugate in the group of all isometries of the Euclidean plane.

4G Coding and Cryptography

Let $\mathcal{A}$ and $\mathcal{B}$ be alphabets of sizes m and a respectively. What does it mean to say that $c : \mathcal{A} \to \mathcal{B}^*$ is a decodable code? State Kraft's inequality.

Suppose that a source emits letters from the alphabet $\mathcal{A} = \{1, 2, \ldots, m\}$, each letter j occurring with (known) probability $p_j > 0$. Let S be the codeword-length random variable for a decodable code $c : \mathcal{A} \to \mathcal{B}^*$, where $|\mathcal{B}| = a$. It is desired to find a decodable code that minimizes the expected value of a^S. Establish the lower bound $\mathbb{E}(a^S) \geqslant (\sum_{j=1}^m \sqrt{p_j})^2$, and characterise when equality occurs. [*Hint. You may use without proof the Cauchy-Schwarz inequality, that (for positive x_i, y_i)*

$$\sum_{i=1}^m x_i y_i \leqslant (\sum_{i=1}^m x_i^2)^{1/2} (\sum_{i=1}^m y_i^2)^{1/2}\,,$$

with equality if and only if $x_i = \lambda y_i$ for all i.]

5K Statistical Modelling

Let $Y_1, \ldots, Y_n$ be independent with $Y_i \sim \frac{1}{n_i}\text{Bin}(n_i, \mu_i)$, $i = 1, \ldots, n$, and

$$\log\left(\frac{\mu_i}{1-\mu_i}\right) = x_i^\top \beta\,, \tag{1}$$

where x_i is a $p \times 1$ vector of regressors and β is a $p \times 1$ vector of parameters. Write down the likelihood of the data $Y_1, \ldots, Y_n$ as a function of $\mu = (\mu_1, \ldots, \mu_n)$. Find the unrestricted maximum likelihood estimator of μ, and the form of the maximum likelihood estimator $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)$ under the logistic model (1).

Show that the *deviance* for a comparison of the full (saturated) model to the generalised linear model with canonical link (1) using the maximum likelihood estimator $\hat{\beta}$ can be simplified to

$$D(y;\hat{\mu}) = -2\sum_{i=1}^{n}\left[n_i y_i x_i^\top \hat{\beta} - n_i \log(1-\hat{\mu}_i)\right].$$

Finally, obtain an expression for the deviance residual in this generalised linear model.

6C Mathematical Biology

Krill is the main food source for baleen whales. The following model has been proposed for the coupled evolution of populations of krill and whales, with $x(t)$ being the number of krill and $y(t)$ being the number of whales:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right) - axy\,,\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{bx}\right),\end{aligned}$$

where r, s, a, b and K are positive constants.

Give a biological interpretation for the form of the two differential equations.

Show that a steady state is possible with $x > 0$ and $y > 0$ and write down expressions for the steady-state values of x and y.

Determine whether this steady state is stable.

7D Dynamical Systems

State the Poincaré–Bendixson theorem.

A model of a chemical process obeys the second-order system

$$\dot{x} = 1 - x(1+a) + x^2 y, \quad \dot{y} = ax - x^2 y\,,$$

where $a > 0$. Show that there is a unique fixed point at $(x, y) = (1, a)$ and that it is unstable if $a > 2$. Show that trajectories enter the region bounded by the lines $x = 1/q$, $y = 0$, $y = aq$ and $x + y = 1 + aq$, provided $q > (1 + a)$. Deduce that there is a periodic orbit when $a > 2$.

8E Further Complex Methods

Recall that if $f(z)$ is analytic in a neighbourhood of $z_0 \neq 0$, then

$$f(z) + \overline{f(z_0)} = 2u\left(\frac{z + \overline{z_0}}{2}, \frac{z - \overline{z_0}}{2i}\right),$$

where $u(x, y)$ is the real part of $f(z)$. Use this fact to construct the imaginary part of an analytic function whose real part is given by

$$u(x, y) = y \int_{-\infty}^{\infty} \frac{g(t)\,dt}{(t - x)^2 + y^2}, \quad x, y \in \mathbb{R},\ y \neq 0\,,$$

where $g(t)$ is real and has sufficient smoothness and decay.

9A Classical Dynamics

Consider a heavy symmetric top of mass M, pinned at point P, which is a distance l from the centre of mass.

(a) Working in the body frame $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ (where $\mathbf{e_3}$ is the symmetry axis of the top) define the *Euler angles* (ψ, θ, ϕ) and show that the components of the angular velocity can be expressed in terms of the Euler angles as

$$\boldsymbol{\omega} = (\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi,\ \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi,\ \dot{\psi} + \dot{\phi} \cos\theta)\,.$$

(b) Write down the Lagrangian of the top in terms of the Euler angles and the principal moments of inertia I_1, I_3.

(c) Find the three constants of motion.

10E Cosmology

The number density of photons in equilibrium at temperature T is given by

$$n = \frac{8\pi}{(hc)^3} \int_0^\infty \frac{\nu^2 d\nu}{e^{\beta h\nu} - 1},$$

where $\beta = 1/(k_B T)$ (k_B is Boltzmann's constant). Show that $n \propto T^3$. Show further that $\epsilon \propto T^4$, where ϵ is the photon energy density.

Write down the Friedmann equation for the scale factor $a(t)$ of a flat homogeneous and isotropic universe. State the relation between a and the mass density ρ for a radiation-dominated universe and hence deduce the time-dependence of a. How does the temperature T depend on time?

SECTION II

11G Geometry and Groups

Define the *axis* of a loxodromic Möbius transformation acting on hyperbolic 3-space.

When do two loxodromic transformations commute? Justify your answer.

Let G be a Kleinian group that contains a loxodromic transformation. Show that the fixed point of any loxodromic transformation in G lies in the limit set of G. Prove that the set of such fixed points is dense in the limit set. Give examples to show that the set of such fixed points can be equal to the limit set or a proper subset.

12G Coding and Cryptography

Define a cyclic binary code of length n.

Show how codewords can be identified with polynomials in such a way that cyclic binary codes correspond to ideals in the polynomial ring with a suitably chosen multiplication rule.

Prove that any cyclic binary code C has a unique generator, that is, a polynomial $c(X)$ of minimum degree, such that the code consists of the multiples of this polynomial. Prove that the rank of the code equals $n - \deg c(X)$, and show that $c(X)$ divides $X^n - 1$.

Show that the repetition and parity check codes are cyclic, and determine their generators.

13K Statistical Modelling

The treatment for a patient diagnosed with cancer of the prostate depends on whether the cancer has spread to the surrounding lymph nodes. It is common to operate on the patient to obtain samples from the nodes which can then be analysed under a microscope. However it would be preferable if an accurate assessment of nodal involvement could be made without surgery. For a sample of 53 prostate cancer patients, a number of possible predictor variables were measured before surgery. The patients then had surgery to determine nodal involvement. We want to see if nodal involvement can be accurately predicted from the available variables and determine which ones are most important. The variables take the values 0 or 1.

`r` An indicator 0=no/1=yes of nodal involvement.

`aged` The patient's age, split into less than 60 (=0) and 60 or over (=1).

`stage` A measurement of the size and position of the tumour observed by palpation with the fingers. A serious case is coded as 1 and a less serious case as 0.

`grade` Another indicator of the seriousness of the cancer which is determined by a pathology reading of a biopsy taken by needle before surgery. A value of 1 indicates a more serious case of cancer.

`xray` Another measure of the seriousness of the cancer taken from an X-ray reading. A value of 1 indicates a more serious case of cancer.

`acid` The level of acid phosphatase in the blood serum where 1=high and 0=low.

A binomial generalised linear model with a logit link was fitted to the data to predict nodal involvement and the following output obtained:

```
Deviance Residuals:
     Min       1Q   Median     3Q     Max
  -2.332  -0.665   -0.300  0.639  2.150

Coefficients:
             Estimate  Std. Error  t value  Pr(>|z|)
(Intercept)   -3.079        0.987    -3.12    0.0018
aged          -0.292        0.754    -0.39    0.6988
grade          0.872        0.816     1.07    0.2850
stage          1.373        0.784     1.75    0.0799
xray           1.801        0.810     2.22    0.0263
acid           1.684        0.791     2.13    0.0334

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 70.252 on 52 degrees of freedom
Residual deviance: 47.611 on 47 degrees of freedom
AIC: 59.61

Number of Fisher Scoring iterations: 5
```

(a) Give an interpretation of the coefficient of `xray`.

(b) Give the numerical value of the sum of the squared deviance residuals.

(c) Suppose that the predictors, `stage`, `grade` and `xray` are positively correlated. Describe the effect that this correlation is likely to have on our ability to determine the strength of these predictors in explaining the response.

(d) The probability of observing a value of 70.252 under a Chi-squared distribution with 52 degrees of freedom is 0.047. What does this information tell us about the null model for this data? Justify your answer.

(e) What is the lowest predicted probability of the nodal involvement for any future patient?

(f) The first plot in Figure 1 shows the (Pearson) residuals and the fitted values. Explain why the points lie on two curves.

(g) The second plot in Figure 1 shows the value of $\hat{\beta} - \hat{\beta}_{(i)}$ where (i) indicates that patient i was dropped in computing the fit. The values for each predictor, including the intercept, are shown. Could a single case change our opinion of which predictors are important in predicting the response?

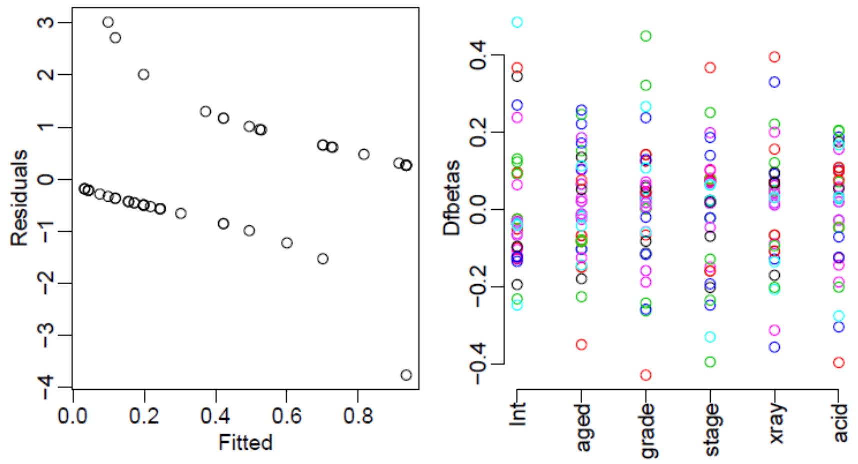

Figure 1: The plot on the left shows the Pearson residuals and the fitted values. The plot on the right shows the changes in the regression coefficients when a single point is omitted for each predictor.

14E Further Complex Methods

(a) Suppose that $F(z),\ z = x + iy,\ x, y \in \mathbb{R}$, is analytic in the upper-half complex z-plane and $O(1/z)$ as $z \to \infty,\ y \geqslant 0$. Show that the real and imaginary parts of $F(x)$, denoted by $U(x)$ and $V(x)$ respectively, satisfy the so-called Kramers–Kronig formulae:

$$U(x) = HV(x), \quad V(x) = -HU(x), \quad x \in \mathbb{R}.$$

Here, H denotes the Hilbert transform, i.e.,

$$(Hf)(x) = \frac{1}{\pi} \, \mathrm{PV} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} \, d\xi ,$$

where PV denotes the principal value integral.

(b) Let the real function $u(x, y)$ satisfy the Laplace equation in the upper-half complex z-plane, i.e.,

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0.$$

Assuming that $u(x, y)$ decays for large $|x|$ and for large y, show that $F = u_z$ is an analytic function for $\operatorname{Im} z > 0,\ z = x + iy$. Then, find an expression for $u_y(x, 0)$ in terms of $u_x(x, 0)$.

15E Cosmology

The Friedmann equation for the scale factor $a(t)$ of a homogeneous and isotropic universe of mass density ρ is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2},$$

where $\dot{a} = da/dt$. Explain how the value of the constant k affects the late-time $(t \to \infty)$ behaviour of a.

Explain briefly why $\rho \propto 1/a^3$ in a matter-dominated (zero-pressure) universe. By considering the scale factor a of a closed universe as a function of conformal time τ, defined by $d\tau = a^{-1}dt$, show that

$$a(\tau) = \frac{\Omega_0}{2(\Omega_0 - 1)}\left[1 - \cos\left(\sqrt{k}c\tau\right)\right],$$

where Ω_0 is the present $(\tau = \tau_0)$ density parameter, with $a(\tau_0) = 1$. Use this result to show that

$$t(\tau) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}\left[\sqrt{k}c\tau - \sin\left(\sqrt{k}c\tau\right)\right],$$

where H_0 is the present Hubble parameter. Find the time t_{BC} at which this model universe ends in a "big crunch".

Given that $\sqrt{k}c\tau_0 \ll 1$, obtain an expression for the present age of the universe in terms of H_0 and Ω_0, according to this model. How does it compare with the age of a flat universe?

16H Logic and Set Theory

State Zorn's lemma, and show how it may be deduced from the Axiom of Choice using the Bourbaki–Witt theorem (which should be clearly stated but not proved).

Show that, if a and b are distinct elements of a distributive lattice L, there is a lattice homomorphism $f : L \to \{0, 1\}$ with $f(a) \neq f(b)$. Indicate briefly how this result may be used to prove the completeness theorem for propositional logic.

17F Graph Theory

State Markov's inequality and Chebyshev's inequality.

Let $\mathcal{G}^{(2)}(n,p)$ denote the probability space of bipartite graphs with vertex classes $U = \{1,2,\ldots,n\}$ and $V = \{-1,-2,\ldots,-n\}$, with each possible edge uv ($u \in U$, $v \in V$) present, independently, with probability p. Let X be the number of subgraphs of $G \in \mathcal{G}^{(2)}(n,p)$ that are isomorphic to the complete bipartite graph $K_{2,2}$. Write down $\mathbb{E}X$ and $\mathrm{Var}\,(X)$. Hence show that $p = 1/n$ is a threshold for $G \in \mathcal{G}^{(2)}(n,p)$ to contain $K_{2,2}$, in the sense that if $np \to \infty$ then a. e. $G \in \mathcal{G}^{(2)}(n,p)$ contains a $K_{2,2}$, whereas if $np \to 0$ then a. e. $G \in \mathcal{G}^{(2)}(n,p)$ does not contain a $K_{2,2}$.

By modifying a random $G \in \mathcal{G}^{(2)}(n,p)$ for suitably chosen p, show that, for each n, there exists a bipartite graph H with n vertices in each class such that $K_{2,2} \not\subset H$ but $e(H) \geqslant \frac{3}{4}\left(\frac{n}{\sqrt[3]{n-1}}\right)^2$.

18H Galois Theory

List all subfields of the cyclotomic field $\mathbb{Q}(\boldsymbol{\mu}_{20})$ obtained by adjoining all 20th roots of unity to $\mathbb{Q}$, and draw the lattice diagram of inclusions among them. Write all the subfields in the form $\mathbb{Q}(\alpha)$ or $\mathbb{Q}(\alpha,\beta)$. Briefly justify your answer.

[The description of the Galois group of cyclotomic fields and the fundamental theorem of Galois theory can be used freely without proof.]

19H Representation Theory

Write down the character table of D_{10}.

Suppose that G is a group of order 60 containing 24 elements of order 5, 20 elements of order 3 and 15 elements of order 2. Calculate the character table of G, justifying your answer.

[You may assume the formula for induction of characters, provided you state it clearly.]

20F Number Fields

Let K be a number field, and $\mathcal{O}_K$ its ring of integers. Write down a characterisation of the units in $\mathcal{O}_K$ in terms of the norm. Without assuming Dirichlet's units theorem, prove that for K a quadratic field the quotient of the unit group by $\{\pm 1\}$ is cyclic (i.e. generated by one element). Find a generator in the cases $K = \mathbb{Q}(\sqrt{-3})$ and $K = \mathbb{Q}(\sqrt{11})$.

Determine all integer solutions of the equation $x^2 - 11y^2 = n$ for $n = -1, 5, 14$.

21G Algebraic Topology

Define the notions of *covering projection* and of *locally path-connected space*. Show that a locally path-connected space is path-connected if it is connected.

Suppose $f\colon Y \to X$ and $g\colon Z \to X$ are continuous maps, the space Y is connected and locally path-connected and that g is a covering projection. Suppose also that we are given base-points x_0, y_0, z_0 satisfying $f(y_0) = x_0 = g(z_0)$. Show that there is a continuous $\tilde{f} : Y \to Z$ satisfying $\tilde{f}(y_0) = z_0$ and $g\tilde{f} = f$ if and only if the image of $f_*\colon \Pi_1(Y, y_0) \to \Pi_1(X, x_0)$ is contained in that of $g_*\colon \Pi_1(Z, z_0) \to \Pi_1(X, x_0)$. [You may assume the path-lifting and homotopy-lifting properties of covering projections.]

Now suppose X is locally path-connected, and both $f\colon Y \to X$ and $g\colon Z \to X$ are covering projections with connected domains. Show that Y and Z are homeomorphic as spaces over X if and only if the images of their fundamental groups under f_* and g_* are conjugate subgroups of $\Pi_1(X, x_0)$.

22G Linear Analysis

What is meant by the *dual* X^* of a normed space X? Show that X^* is a Banach space.

Let $X = C^1(0,1)$, the space of functions $f : (0,1) \to \mathbb{R}$ possessing a bounded, continuous first derivative. Endow X with the sup norm $\|f\|_\infty = \sup_{x\in(0,1)} |f(x)|$. Which of the following maps $T : X \to \mathbb{R}$ are elements of X^*? Give brief justifications or counterexamples as appropriate.

1. $Tf = f(\frac{1}{2})$;
2. $Tf = \|f\|_\infty$;
3. $Tf = \int_0^1 f(x)\,dx$;
4. $Tf = f'(\frac{1}{2})$.

Now suppose that X is a (real) Hilbert space. State and prove the Riesz representation theorem. Describe the natural map $X \to X^{**}$ and show that it is surjective.

[All normed spaces are over $\mathbb{R}$. You may assume that if Y is a closed subspace of a Hilbert space X then $X = Y \oplus Y^\perp$.]

23I Riemann Surfaces

(i) Let $f(z) = \sum_{n=1}^{\infty} z^{2^n}$. Show that the unit circle is the natural boundary of the function element $(D(0,1), f)$.

(ii) Let $U = \{z \in \mathbf{C} \ : \ \mathrm{Re}(z) > 0\} \subset \mathbf{C}$; explain carefully how a holomorphic function f may be defined on U satisfying the equation

$$(f(z)^2 - 1)^2 = z\,.$$

Let $\mathcal{F}$ denote the connected component of the space of germs $\mathcal{G}$ (of holomorphic functions on $\mathbf{C} \setminus \{0\}$) corresponding to the function element (U, f), with associated holomorphic map $\pi : \mathcal{F} \to \mathbf{C} \setminus \{0\}$. Determine the number of points of $\mathcal{F}$ in $\pi^{-1}(w)$ when (a) $w = \frac{1}{2}$, and (b) $w = 1$.

[You may assume any standard facts about analytic continuations that you may need.]

24I Algebraic Geometry

(a) Let X be an affine variety, $k[X]$ its ring of functions, and let $p \in X$. Assume k is algebraically closed. Define the *tangent space* T_pX at p. Prove the following assertions.

(i) A morphism of affine varieties $f : X \to Y$ induces a linear map

$$df : T_pX \to T_{f(p)}Y.$$

(ii) If $g \in k[X]$ and $U := \{x \in X \mid g(x) \neq 0\}$, then U has the natural structure of an affine variety, and the natural morphism of U into X induces an isomorphism $T_pU \to T_pX$ for all $p \in U$.

(iii) For all $s \geqslant 0$, the subset $\{x \in X \mid \dim T_xX \geqslant s\}$ is a Zariski-closed subvariety of X.

(b) Show that the set of nilpotent 2×2 matrices

$$X = \{x \in \mathrm{Mat}_2(k) \,|\, x^2 = 0\}$$

may be realised as an affine surface in $\mathbf{A}^3$, and determine its tangent space at all points $x \in X$.

Define what it means for two varieties Y_1 and Y_2 to be *birationally equivalent*, and show that the variety X of nilpotent 2×2 matrices is birationally equivalent to $\mathbf{A}^2$.

25I Differential Geometry

Define the *geodesic curvature* k_g of a regular curve in an oriented surface $S \subset \mathbb{R}^3$. When is $k_g = 0$ along a curve?

Explain briefly what is meant by the *Euler characteristic* χ of a compact surface $S \subset \mathbb{R}^3$. State the global Gauss–Bonnet theorem with boundary terms.

Let S be a surface with positive Gaussian curvature that is diffeomorphic to the sphere S^2 and let γ_1,γ_2 be two disjoint simple closed curves in S. Can both γ_1 and γ_2 be geodesics? Can both γ_1 and γ_2 have constant geodesic curvature? Justify your answers.

[You may assume that the complement of a simple closed curve in S^2 consists of two open connected regions.]

26J Probability and Measure

Carefully state and prove Jensen's inequality for a convex function $c : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval. Assuming that c is strictly convex, give necessary and sufficient conditions for the inequality to be strict.

Let μ be a Borel probability measure on $\mathbb{R}$, and suppose μ has a strictly positive probability density function f_0 with respect to Lebesgue measure. Let $\mathcal{P}$ be the family of all strictly positive probability density functions f on $\mathbb{R}$ with respect to Lebesgue measure such that $\log(f/f_0) \in L^1(\mu)$. Let X be a random variable with distribution μ. Prove that the mapping

$$f \mapsto \mathbb{E}\left[\log \frac{f}{f_0}(X)\right]$$

has a unique maximiser over $\mathcal{P}$, attained when $f = f_0$ almost everywhere.

27K Applied Probability

(a) Give the definition of a *Poisson process* $(N_t, t \geqslant 0)$ with rate λ, using its transition rates. Show that for each $t \geqslant 0$, the distribution of N_t is Poisson with a parameter to be specified.

Let $J_0 = 0$ and let $J_1, J_2, \ldots$ denote the jump times of $(N_t, t \geqslant 0)$. What is the distribution of $(J_{n+1} - J_n,\ n \geqslant 0)$? (You do not need to justify your answer.)

(b) Let $n \geqslant 1$. Compute the joint probability density function of $(J_1, J_2, \ldots, J_n)$ given $\{N_t = n\}$. Deduce that, given $\{N_t = n\}$, $(J_1, \ldots, J_n)$ has the same distribution as the nondecreasing rearrangement of n independent uniform random variables on $[0, t]$.

(c) Starting from time 0, passengers arrive on platform 9B at King's Cross station, with constant rate $\lambda > 0$, in order to catch a train due to depart at time $t > 0$. Using the above results, or otherwise, find the expected total time waited by all passengers (the sum of all passengers' waiting times).

28K Principles of Statistics

Prove that, if T is complete sufficient for Θ, and S is a function of T, then S is the minimum variance unbiased estimator of $\mathbb{E}(S \mid \Theta)$.

When the parameter Θ takes a value $\theta > 0$, observables $(X_1, \ldots, X_n)$ arise independently from the exponential distribution $\mathcal{E}(\theta)$, having probability density function

$$p(x \mid \theta) = \theta e^{-\theta x} \quad (x > 0)\,.$$

Show that the family of distributions

$$\Theta \sim \text{Gamma}\ (\alpha, \beta) \quad (\alpha > 0, \beta > 0)\,, \tag{1}$$

with probability density function

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \quad (\theta > 0)\,,$$

is a conjugate family for Bayesian inference about Θ (where $\Gamma(\alpha)$ is the Gamma function).

Show that the expectation of $\Lambda := \log \Theta$, under prior distribution (1), is $\psi(\alpha) - \log \beta$, where $\psi(\alpha) := (\mathrm{d}/\mathrm{d}\alpha) \log \Gamma(\alpha)$. What is the prior variance of Λ? Deduce the posterior expectation and variance of Λ, given $(X_1, \ldots, X_n)$.

Let $\widetilde{\Lambda}$ denote the limiting form of the posterior expectation of Λ as $\alpha, \beta \downarrow 0$. Show that $\widetilde{\Lambda}$ is the minimum variance unbiased estimator of Λ. What is its variance?

29J Stochastic Financial Models

Consider a multi-period binomial model with a risky asset $(S_0, \ldots, S_T)$ and a riskless asset $(B_0, \ldots, B_T)$. In each period, the value of the risky asset S is multiplied by u if the period was good, and by d otherwise. The riskless asset is worth $B_t = (1 + r)^t$ at time $0 \leqslant t \leqslant T$, where $r \geqslant 0$.

(i) Assuming that $T = 1$ and that

$$d < 1 + r < u\,, \tag{1}$$

show how any contingent claim to be paid at time 1 can be priced and exactly replicated. Briefly explain the significance of the condition (1), and indicate how the analysis of the single-period model extends to many periods.

(ii) Now suppose that $T = 2$. We assume that $u = 2$, $d = 1/3$, $r = 1/2$, and that the risky asset is worth $S_0 = 27$ at time zero. Find the time-0 value of an American put option with strike price $K = 28$ and expiry at time $T = 2$, and find the optimal exercise policy. (Assume that the option cannot be exercised immediately at time zero.)

30B Partial Differential Equations
Let $u_0 : \mathbb{R} \to \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$, $u_0(x) \geqslant 0$ for all $x \in \mathbb{R}$. Consider the partial differential equation for $u = u(x,y)$,

$$4yu_x + 3u_y = u^2, \qquad (x,y) \in \mathbb{R}^2$$

subject to the Cauchy condition $u(x,0) = u_0(x)$.

i) Compute the solution of the Cauchy problem by the method of characteristics.

ii) Prove that the domain of definition of the solution contains

$$(x,y) \in \mathbb{R} \times \left(-\infty, \frac{3}{\sup_{x\in\mathbb{R}}\left(u_0(x)\right)}\right).$$

31B Asymptotic Methods
What precisely is meant by the statement that

$$f(x) \sim \sum_{n=0}^{\infty} d_n\, x^n \qquad (*)$$

as $x \to 0$?

Consider the Stieltjes integral

$$I(x) = \int_1^{\infty} \frac{\rho(t)}{1+xt} dt\,,$$

where $\rho(t)$ is bounded and decays rapidly as $t \to \infty$, and $x > 0$. Find an asymptotic series for $I(x)$ of the form $(*)$, as $x \to 0$, and prove that it has the asymptotic property.

In the case that $\rho(t) = e^{-t}$, show that the coefficients d_n satisfy the recurrence relation

$$d_n = (-1)^n \frac{1}{e} - n\, d_{n-1} \qquad (n \geqslant 1)$$

and that $d_0 = \dfrac{1}{e}$. Hence find the first three terms in the asymptotic series.

32D Integrable Systems

State the Arnold–Liouville theorem.

Consider an integrable system with six-dimensional phase space, and assume that $\nabla \wedge \mathbf{p} = 0$ on any Liouville tori $p_i = p_i(q_j, c_j)$, where $\nabla = (\partial/\partial q_1, \partial/\partial q_2, \partial/\partial q_3)$.

(a) Define the action variables and use Stokes' theorem to show that the actions are independent of the choice of the cycles.

(b) Define the generating function, and show that the angle coordinates are periodic with period 2π.

33A Principles of Quantum Mechanics

Let a and $a^\dagger$ be the simple harmonic oscillator annihilation and creation operators, respectively. Write down the commutator $[a,\ a^\dagger]$.

Consider a new operator $b = ca + sa^\dagger$, where $c \equiv \cosh\theta$, $s \equiv \sinh\theta$ with θ a real constant. Show that

$$[b, b^\dagger] = 1.$$

Consider the Hamiltonian

$$H = \epsilon a^\dagger a + \frac{1}{2}\lambda({a^\dagger}^2 + a^2)\,,$$

where ϵ and λ are real and such that $\epsilon > \lambda > 0$. Assuming that $\epsilon c - \lambda s = Ec$ and $\lambda c - \epsilon s = Es$, with E a real constant, show that

$$[b, H] = Eb\,.$$

Thus, calculate the energy of $b|E_a\rangle$ in terms of E and E_a, where E_a is an eigenvalue of H.

Assuming that $b|E_{\text{min}}\rangle = 0$, calculate E_{min} in terms of λ, s and c. Find the possible values of $x = s/c$. Finally, show that the energy eigenvalues of the system are

$$E_n = -\frac{\epsilon}{2} + (n + \frac{1}{2})\sqrt{\epsilon^2 - \lambda^2}\,.$$

34E Applications of Quantum Mechanics

Give an account of the variational principle for establishing an upper bound on the ground-state energy E_0 of a particle moving in a potential $V(x)$ in one dimension.

A particle of unit mass moves in the potential

$$V(x) \;=\; \begin{cases} \infty & x \leqslant 0 \\ \lambda x & x > 0 \end{cases} \,,$$

with λ a positive constant. Explain why it is important that any trial wavefunction used to derive an upper bound on E_0 should be chosen to vanish for $x \leqslant 0$.

Use the trial wavefunction

$$\psi(x) \;=\; \begin{cases} 0 & x \leqslant 0 \\ x e^{-ax} & x > 0 \end{cases} \,,$$

where a is a positive real parameter, to establish an upper bound $E_0 \leqslant E(a, \lambda)$ for the energy of the ground state, and hence derive the lowest upper bound on E_0 as a function of λ.

Explain why the variational method cannot be used in this case to derive an upper bound for the energy of the first excited state.

35C Statistical Physics

A meson consists of two quarks, attracted by a linear potential energy

$$V = \alpha x \,,$$

where x is the separation between the quarks and α is a constant. Treating the quarks classically, compute the vibrational partition function that arises from the separation of quarks. What is the average separation of the quarks at temperature T?

Consider an ideal gas of these mesons that have the orientation of the quarks fixed so the mesons do not rotate. Compute the total partition function of the gas. What is its heat capacity C_V?

[Note: $\int_{-\infty}^{+\infty} dx \; e^{-ax^2} = \sqrt{\pi/a}$.]

36B Electrodynamics

A particle of mass m and charge q moves relativistically under the influence of a constant electric field E in the positive z-direction, and a constant magnetic field B also in the positive z-direction.

In some inertial observer's coordinate system, the particle starts at

$$x = R, \quad y = 0, \quad z = 0, \quad t = 0,$$

with velocity given by

$$\dot{x} = 0, \quad \dot{y} = u, \quad \dot{z} = 0,$$

where the dot indicates differentiation with respect to the proper time of the particle. Show that the subsequent motion of the particle, as seen by the inertial observer, is a helix.

a) What is the radius of the helix as seen by the inertial observer?

b) What are the x and y coordinates of the axis of the helix?

c) What is the z coordinate of the particle after a proper time τ has elapsed, as measured by the particle?

37B General Relativity

(i) Using the condition that the metric tensor g_{ab} is covariantly constant, derive an expression for the Christoffel symbol $\Gamma^a_{bc} = \Gamma^a_{cb}$.

(ii) Show that

$$\Gamma^a_{ba} = \frac{1}{2} g^{ac} g_{ac,b} \,.$$

Hence establish the covariant divergence formula

$$V^a{}_{;a} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} \left(\sqrt{-g}\, V^a \right) \,,$$

where g is the determinant of the metric tensor.

[It may be assumed that $\partial_a(\log \det M) = \mathrm{trace}\,(M^{-1}\partial_a M)$ for any invertible matrix M].

(iii) The Kerr-Newman metric, describing the spacetime outside a rotating black hole of mass M, charge Q and angular momentum per unit mass a, is given in appropriate units by

$$\begin{aligned} ds^2 = &- (dt - a \sin^2\theta \, d\phi)^2 \frac{\Delta}{\rho^2} \\ &+ \left((r^2 + a^2) d\phi - a\, dt \right)^2 \frac{\sin^2\theta}{\rho^2} + \left(\frac{dr^2}{\Delta} + d\theta^2 \right) \rho^2 \,, \end{aligned}$$

where $\rho^2 = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2 + Q^2$. Explain why this metric is stationary, and make a choice of one of the parameters which reduces it to a static metric.

Show that, in the static metric obtained, the equation

$$(g^{ab} \Phi_{,b})_{;a} = 0$$

for a function $\Phi = \Phi(t, r)$ admits solutions of the form

$$\Phi = \sin(\omega t) R(r) \,,$$

where ω is constant and $R(r)$ satisfies an ordinary differential equation which should be found.

38C Fluid Dynamics II

Define the strain-rate tensor e_{ij} in terms of the velocity components u_i. Write down the relation between e_{ij}, the pressure p and the stress σ_{ij} in an incompressible Newtonian fluid of viscosity μ. Show that the local rate of stress-working $\sigma_{ij}\partial u_i/\partial x_j$ is equal to the local rate of dissipation $2\mu e_{ij}e_{ij}$.

An incompressible fluid of density ρ and viscosity μ occupies the semi-infinite region $y > 0$ above a rigid plane boundary $y = 0$ which oscillates with velocity $(V\cos\omega t, 0, 0)$. The fluid is at rest at infinity. Determine the velocity field produced by the boundary motion after any transients have decayed.

Show that the time-averaged rate of dissipation is

$$\tfrac{1}{4}\sqrt{2}V^2\,(\mu\rho\omega)^{1/2}$$

per unit area of the boundary. Verify that this is equal to the time average of the rate of working by the boundary on the fluid per unit area.

39D Waves

Write down the linearized equations governing motion in an inviscid compressible fluid and, assuming an adiabatic relationship $p = p(\rho)$, derive the wave equation for the velocity potential $\phi(\mathbf{x}, t)$. Obtain from these linearized equations the energy equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{I} = 0\,,$$

and give expressions for the acoustic energy density E and the acoustic intensity, or energy-flux vector, $\mathbf{I}$.

An inviscid compressible fluid occupies the half-space $y > 0$, and is bounded by a very thin flexible membrane of negligible mass at an undisturbed position $y = 0$. Small acoustic disturbances with velocity potential $\phi(x, y, t)$ in the fluid cause the membrane to be deflected to $y = \eta(x, t)$. The membrane is supported by springs that, in the deflected state, exert a restoring force $K\eta\,\delta x$ on an element δx of the membrane. Show that the dispersion relation for waves proportional to $\exp(ikx - i\omega t)$ propagating freely along the membrane is

$$\left(k^2 - \frac{\omega^2}{c_0^2}\right)^{1/2} - \frac{\rho_0\omega^2}{K} = 0\,,$$

where ρ_0 is the density of the fluid and c_0 is the sound speed. Show that in such a wave the component $\langle I_y\rangle$ of mean acoustic intensity perpendicular to the membrane is zero.

40D Numerical Analysis

The Poisson equation $u_{xx} = f$ in the unit interval $\Omega = [0,1]$, $u = 0$ on $\partial\Omega$ is discretised with the formula

$$u_{i-1} + u_{i+1} - 2u_i = h^2 f_i\,,$$

where $1 \leqslant i \leqslant n$, $u_i \approx u(ih)$ and ih are the grid points.

(i) Define the above system of equations in vector form $A\mathbf{u} = \mathbf{b}$ and describe the relaxed Jacobi method with relaxation parameter ω for solving this linear system. For $\mathbf{x}^*$ and $\mathbf{x}^{(\nu)}$ being the exact solution and the iterated solution respectively, let $\mathbf{e}^{(\nu)} = \mathbf{x}^{(\nu)} - \mathbf{x}^*$ be the error and H_ω the iteration matrix, so that

$$\mathbf{e}^{(\nu+1)} = H_\omega \mathbf{e}^{(\nu)}\,.$$

Express H_ω in terms of the matrix A, the diagonal part D of A and ω, and find the eigenvectors $\mathbf{v}_k$ and the eigenvalues $\lambda_k(\omega)$ of H_ω.

(ii) For A as above, let

$$\mathbf{e}^{(\nu)} = \sum_{k=1}^{n} a_k^{(\nu)} \mathbf{v}_k$$

be the expansion of the error with respect to the eigenvectors of H_ω. Derive conditions on ω such that the method converges for any n, and prove that, for any such ω, the rate of convergence of $\mathbf{e}^{(\nu)} \to 0$ is not faster than $(1 - c/n^2)^\nu$.

(iii) Show that, for some ω, the high frequency components ($\frac{n+1}{2} \leqslant k \leqslant n$) of the error $\mathbf{e}^{(\nu)}$ tend to zero much faster than $(1 - c/n^2)^\nu$. Determine the optimal parameter ω_* which provides the largest suppression of the high frequency components per iteration, and find the corresponding attenuation factor μ_* (i.e., the least μ_ω such that $|a_k^{(\nu+1)}| \leqslant \mu_\omega |a_k^{(\nu)}|$ for $\frac{n+1}{2} \leqslant k \leqslant n$).

END OF PAPER