UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 6 June, 2019 1:30 pm to 4:30 pm MAT1

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

Prove that the ideal $(2, 1+\sqrt{-13})$ in $\mathbb{Z}[\sqrt{-13}]$ is not principal.

2E Analysis II

(a) Let $A \subset \mathbb{R}$. What does it mean for a function $f : A \to \mathbb{R}$ to be *uniformly continuous*?

(b) Which of the following functions are uniformly continuous? Briefly justify your answers.

(i) $f(x) = x^2$ on $\mathbb{R}$.

(ii) $f(x) = \sqrt{x}$ on $[0, \infty)$.

(iii) $f(x) = \cos(1/x)$ on $[1, \infty)$.

3G Metric and Topological Spaces

Let X be a metric space.

(a) What does it mean for X to be *compact*? What does it mean for X to be *sequentially compact*?

(b) Prove that if X is compact then X is sequentially compact.

4D Complex Methods

By considering the transformation $w = i(1-z)/(1+z)$, find a solution to Laplace's equation $\nabla^2\phi = 0$ inside the unit disc $D \subset \mathbb{C}$, subject to the boundary conditions

$$\phi\Big|_{|z|=1} = \begin{cases} \phi_0 & \text{for } \arg(z) \in (0, \pi) \\ -\phi_0 & \text{for } \arg(z) \in (\pi, 2\pi)\,, \end{cases}$$

where ϕ_0 is constant. Give your answer in terms of $(x, y) = (\operatorname{Re} z, \operatorname{Im} z)$.

5E Geometry

State a formula for the area of a spherical triangle with angles α, β, γ.

Let $n \geqslant 3$. What is the area of a convex spherical n-gon with interior angles $\alpha_1, \ldots, \alpha_n$? Justify your answer.

Find the range of possible values for the interior angle of a regular convex spherical n-gon.

6A Variational Principles

The function f with domain $x > 0$ is defined by $f(x) = \frac{1}{a}x^a$, where $a > 1$. Verify that f is convex, using an appropriate sufficient condition.

Determine the Legendre transform f^* of f, specifying clearly its domain of definition, and find $(f^*)^*$.

Show that

$$\frac{x^r}{r} + \frac{y^s}{s} \geqslant xy$$

where x, $y > 0$ and r and s are positive real numbers such that $\frac{1}{r} + \frac{1}{s} = 1$.

7D Methods

Define the *discrete Fourier transform* of a sequence $\{x_0, x_1, \ldots, x_{N-1}\}$ of N complex numbers.

Compute the discrete Fourier transform of the sequence

$$x_n = \frac{1}{N}\left(1 + e^{2\pi i n/N}\right)^{N-1} \qquad \text{for} \quad n = 0, \ldots, N-1\,.$$

8B Quantum Mechanics

Consider a quantum mechanical particle moving in two dimensions with Cartesian coordinates x, y. Show that, for wavefunctions with suitable decay as $x^2 + y^2 \to \infty$, the operators

$$x \quad \text{and} \quad -i\hbar\frac{\partial}{\partial x}$$

are Hermitian, and similarly

$$y \quad \text{and} \quad -i\hbar\frac{\partial}{\partial y}$$

are Hermitian.

Show that if F and G are Hermitian operators, then

$$\frac{1}{2}(FG + GF)$$

is Hermitian. Deduce that

$$L = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad \text{and} \quad D = -i\hbar\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 1\right)$$

are Hermitian. Show that

$$[L, D] = 0.$$

9H Markov Chains

Suppose that (X_n) is a Markov chain with state space S.

(a) Give the definition of a *communicating class*.

(b) Give the definition of the *period* of a state $a \in S$.

(c) Show that if two states communicate then they have the same period.

SECTION II

10F Linear Algebra

If q is a quadratic form on a finite-dimensional real vector space V, what is the associated *symmetric bilinear form* $\varphi(\cdot,\cdot)$? Prove that there is a basis for V with respect to which the matrix for φ is diagonal. What is the *signature* of q?

If $R \leqslant V$ is a subspace such that $\varphi(r, v) = 0$ for all $r \in R$ and all $v \in V$, show that $q'(v + R) = q(v)$ defines a quadratic form on the quotient vector space V/R. Show that the signature of q' is the same as that of q.

If $e, f \in V$ are vectors such that $\varphi(e, e) = 0$ and $\varphi(e, f) = 1$, show that there is a direct sum decomposition $V = \operatorname{span}(e, f) \oplus U$ such that the signature of $q|_U$ is the same as that of q.

11G Groups, Rings and Modules

Let $\omega = \frac{1}{2}(-1 + \sqrt{-3})$.

(a) Prove that $\mathbb{Z}[\omega]$ is a Euclidean domain.

(b) Deduce that $\mathbb{Z}[\omega]$ is a unique factorisation domain, stating carefully any results from the course that you use.

(c) By working in $\mathbb{Z}[\omega]$, show that whenever $x, y \in \mathbb{Z}$ satisfy

$$x^2 - x + 1 = y^3$$

then x is not congruent to 2 modulo 3.

12E Analysis II

(a) Carefully state the Picard–Lindelöf theorem on solutions to ordinary differential equations.

(b) Let $X = C([1,b],\mathbb{R}^n)$ be the set of continuous functions from a closed interval $[1,b]$ to $\mathbb{R}^n$, and let $||\cdot||$ be a norm on $\mathbb{R}^n$.

(i) Let $f \in X$. Show that for any $c \in [0,\infty)$ the norm

$$||f||_c = \sup_{t\in[1,b]} ||f(t)t^{-c}||$$

is Lipschitz equivalent to the usual sup norm on X.

(ii) Assume that $F : [1,b]\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and Lipschitz in the second variable, i.e. there exists $M > 0$ such that

$$\|F(t,x) - F(t,y)\| \leqslant M\|x-y\|$$

for all $t \in [1,b]$ and all $x, y \in \mathbb{R}^n$. Define $\varphi : X \to X$ by

$$\varphi(f)(t) = \int_1^t F(l, f(l))\,dl$$

for $t \in [1,b]$.

Show that there is a choice of c such that φ is a contraction on $(X, ||\cdot||_c)$. Deduce that for any $y_0 \in \mathbb{R}^n$, the differential equation

$$Df(t) = F(t, f(t))$$

has a unique solution on $[1,b]$ with $f(1) = y_0$.

13F Complex Analysis

Define the *winding number* $n(\gamma, w)$ of a closed path $\gamma : [a,b] \to \mathbb{C}$ around a point $w \in \mathbb{C}$ which does not lie on the image of γ. [You do not need to justify its existence.]

If f is a meromorphic function, define the *order* of a zero z_0 of f and of a pole w_0 of f. State the Argument Principle, and explain how it can be deduced from the Residue Theorem.

How many roots of the polynomial

$$z^4 + 10z^3 + 4z^2 + 10z + 5$$

lie in the right-hand half plane?

14E Geometry

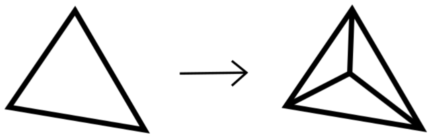
Define a *geodesic triangulation* of an abstract closed smooth surface. Define the *Euler number* of a triangulation, and state the Gauss–Bonnet theorem for closed smooth surfaces. Given a vertex in a triangulation, its valency is defined to be the number of edges incident at that vertex.

(a) Given a triangulation of the torus, show that the average valency of a vertex of the triangulation is 6.

(b) Consider a triangulation of the sphere.

(i) Show that the average valency of a vertex is strictly less than 6.

(ii) A triangulation can be subdivided by replacing one triangle Δ with three sub-triangles, each one with vertices two of the original ones, and a fixed interior point of Δ.



Using this, or otherwise, show that there exist triangulations of the sphere with average vertex valency arbitrarily close to 6.

(c) Suppose S is a closed abstract smooth surface of everywhere negative curvature. Show that the average vertex valency of a triangulation of S is bounded above and below.

15D Methods

By differentiating the expression $\psi(t) = H(t)\,\sin(\alpha t)/\alpha$, where α is a constant and $H(t)$ is the Heaviside step function, show that

$$\frac{d^2\psi}{dt^2} + \alpha^2\psi = \delta(t)\,,$$

where $\delta(t)$ is the Dirac δ-function.

Hence, by taking a Fourier transform with respect to the spatial variables only, derive the retarded Green's function for the wave operator $\partial_t^2 - c^2\nabla^2$ in three spatial dimensions.

[*You may use that*

$$\frac{1}{2\pi}\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{\sin(kct)}{kc}\,d^3k = -\frac{i}{c|\mathbf{x}-\mathbf{y}|}\int_{-\infty}^{\infty} e^{ik|\mathbf{x}-\mathbf{y}|}\,\sin(kct)\,dk$$

without proof.]

Thus show that the solution to the homogeneous wave equation $\partial_t^2 u - c^2\nabla^2 u = 0$, subject to the initial conditions $u(\mathbf{x},0) = 0$ and $\partial_t u(\mathbf{x},0) = f(\mathbf{x})$, may be expressed as

$$u(\mathbf{x},t) = \langle f\rangle\, t\,,$$

where $\langle f\rangle$ is the average value of f on a sphere of radius ct centred on $\mathbf{x}$. Interpret this result.

16B Quantum Mechanics

Consider a particle of unit mass in a one-dimensional square well potential

$$V(x) = 0 \quad \text{for} \quad 0 \leqslant x \leqslant \pi\,,$$

with V infinite outside. Find all the stationary states $\psi_n(x)$ and their energies E_n, and write down the general normalized solution of the time-dependent Schrödinger equation in terms of these.

The particle is initially constrained by a barrier to be in the ground state in the narrower square well potential

$$V(x) = 0 \quad \text{for} \quad 0 \leqslant x \leqslant \frac{\pi}{2}\,,$$

with V infinite outside. The barrier is removed at time $t = 0$, and the wavefunction is instantaneously unchanged. Show that the particle is now in a superposition of stationary states of the original potential well, and calculate the probability that an energy measurement will yield the result E_n.

17A Electromagnetism

The electric and magnetic fields $\mathbf{E}$, $\mathbf{B}$ in an inertial frame $\mathcal{S}$ are related to the fields $\mathbf{E}'$, $\mathbf{B}'$ in a frame $\mathcal{S}'$ by a Lorentz transformation. Given that $\mathcal{S}'$ moves in the x-direction with speed v relative to $\mathcal{S}$, and that

$$E'_y = \gamma(E_y - vB_z)\,, \quad B'_z = \gamma(B_z - (v/c^2)E_y)\,,$$

write down equations relating the remaining field components and define γ. Use your answers to show directly that $\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E} \cdot \mathbf{B}$.

Give an expression for an additional, independent, Lorentz-invariant function of the fields, and check that it is invariant for the special case when $E_y = E$ and $B_y = B$ are the only non-zero components in the frame $\mathcal{S}$.

Now suppose in addition that $cB = \lambda E$ with λ a non-zero constant. Show that the angle θ between the electric and magnetic fields in $\mathcal{S}'$ is given by

$$\cos\theta \;=\; f(\beta) \;=\; \frac{\lambda(1-\beta^2)}{\{(1+\lambda^2\beta^2)(\lambda^2+\beta^2)\}^{1/2}}$$

where $\beta = v/c$. By considering the behaviour of $f(\beta)$ as β approaches its limiting values, show that the relative velocity of the frames can be chosen so that the angle takes any value in one of the ranges $0 \leqslant \theta < \pi/2$ or $\pi/2 < \theta \leqslant \pi$, depending on the sign of λ.

18C Fluid Dynamics

A cubic box of side $2h$, enclosing the region $0 < x < 2h$, $0 < y < 2h$, $-h < z < h$, contains equal volumes of two incompressible fluids that remain distinct. The system is initially at rest, with the fluid of density ρ_1 occupying the region $0 < z < h$ and the fluid of density ρ_2 occupying the region $-h < z < 0$, and with gravity $(0, 0, -g)$. The interface between the fluids is then slightly perturbed. Derive the linearized equations and boundary conditions governing small disturbances to the initial state.

In the case $\rho_2 > \rho_1$, show that the angular frequencies ω of the normal modes are given by

$$\omega^2 = \left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) gk \tanh(kh)$$

and express the allowable values of the wavenumber k in terms of h. Identify the lowest-frequency non-trivial mode(s). Comment on the limit $\rho_1 \ll \rho_2$. What physical behaviour is expected in the case $\rho_1 > \rho_2$?

19C Numerical Analysis

(a) Let $w(x)$ be a positive weight function on the interval $[a,b]$. Show that

$$\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\,dx$$

defines an inner product on $C[a,b]$.

(b) Consider the sequence of polynomials $p_n(x)$ defined by the three-term recurrence relation

$$p_{n+1}(x) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x)\,, \qquad n = 1,2,\ldots\,, \qquad (*)$$

where

$$p_0(x) = 1\,, \qquad p_1(x) = x - \alpha_0\,,$$

and the coefficients α_n (for $n \geqslant 0$) and β_n (for $n \geqslant 1$) are given by

$$\alpha_n = \frac{\langle p_n, xp_n\rangle}{\langle p_n,p_n\rangle}\,, \qquad \beta_n = \frac{\langle p_n,p_n\rangle}{\langle p_{n-1},p_{n-1}\rangle}\,.$$

Prove that this defines a sequence of monic orthogonal polynomials on $[a,b]$.

(c) The Hermite polynomials $He_n(x)$ are orthogonal on the interval $(-\infty,\infty)$ with weight function $e^{-x^2/2}$. Given that

$$He_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)\,,$$

deduce that the Hermite polynomials satisfy a relation of the form $(*)$ with $\alpha_n = 0$ and $\beta_n = n$. Show that $\langle He_n, He_n\rangle = n!\sqrt{2\pi}$.

(d) State, without proof, how the properties of the Hermite polynomial $He_N(x)$, for some positive integer N, can be used to estimate the integral

$$\int_{-\infty}^{\infty} f(x)\,e^{-x^2/2}\,dx\,,$$

where $f(x)$ is a given function, by the method of Gaussian quadrature. For which polynomials is the quadrature formula exact?

20H Statistics

Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$. Let

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_{XX} = \sum_{i=1}^{n}(X_i - \overline{X})^2 .$$

(a) Compute the distributions of $\overline{X}$ and S_{XX} and show that $\overline{X}$ and S_{XX} are independent.

(b) Write down the distribution of $\sqrt{n}(\overline{X} - \mu)/\sqrt{S_{XX}(n-1)}$.

(c) For $\alpha \in (0,1)$, find a $100(1-\alpha)\%$ confidence interval in each of the following situations:

(i) for μ when σ^2 is known;

(ii) for μ when σ^2 is not known;

(iii) for σ^2 when μ is not known.

(d) Suppose that $\widetilde{X}_1, \ldots, \widetilde{X}_{\widetilde{n}}$ are i.i.d. $N(\widetilde{\mu}, \widetilde{\sigma}^2)$. Explain how you would use the F-test to test the hypothesis $H_1 : \sigma^2 > \widetilde{\sigma}^2$ against the hypothesis $H_0 : \sigma^2 = \widetilde{\sigma}^2$. Does the F-test depend on whether μ, $\widetilde{\mu}$ are known?

21H Optimisation

(a) Suppose that $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, with $n \geqslant m$. What does it mean for $x \in \mathbb{R}^n$ to be a *basic feasible solution* of the equation $Ax = b$?

Assume that the m rows of A are linearly independent, every set of m columns is linearly independent, and every basic solution has exactly m non-zero entries. Prove that the extreme points of $\mathcal{X}(b) = \{x \geqslant 0 : Ax = b\}$ are the basic feasible solutions of $Ax = b$. [Here, $x \geqslant 0$ means that each of the coordinates of x are at least 0.]

(b) Use the simplex method to solve the linear program

$$\begin{aligned}
\text{max} \quad & 4x_1 + 3x_2 + 7x_3 \\
\text{s.t.} \quad & x_1 + 3x_2 + x_3 \leqslant 14 \\
& 4x_1 + 3x_2 + 2x_3 \leqslant 5 \\
& -x_1 + x_2 - x_3 \geqslant -2 \\
& x_1, x_2, x_3 \geqslant 0.
\end{aligned}$$

END OF PAPER