UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2024

List of Courses

Paper 2, Section I

2F Analysis and Topology

Let (X, d) be a metric space. Define what it means for $h : X \to X$ to be a *contraction*.

State and prove the contraction mapping theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, and let r be a root of f. Suppose that on some neighbourhood U of r, $|f'(x)| > \delta$ for some $\delta > 0$ and $|f''(x)| < M$ for some $M < \infty$. Define $g : U \to \mathbb{R}$ by $g(x) = x - f(x)/f'(x)$. Show that $g'(r) = 0$ and that g' is bounded by $1/2$ in absolute value on some neighbourhood U' of r. Deduce that r is the unique fixed point of g on U'.

Paper 4, Section I

2F Analysis and Topology

Define what it means for a subset A of a topological space (X, τ) to be *connected*.

Let $f : X \to Y$ be a continuous map between topological spaces (X, τ) and (Y, σ). Show that if X is connected, then $f(X)$ is connected.

Let $Y = \{0, 1\}$ be equipped with the discrete topology. Show that a topological space (X, τ) is connected if and only if every continuous function $h : X \to Y$ is constant.

Given a subset A of a topological space (X, τ), define the *closure* $\mathrm{Cl}(A)$ of A to be the set of all points of A together with the set of points $y \in X$ such that every open set in τ containing y contains some point of A other than y. Using the preceding part or otherwise, show that given a connected set $C \subseteq X$, $\mathrm{Cl}(C)$ is connected.

Paper 1, Section II
10F Analysis and Topology

(a) Define what it means for a metric space (X,d) to be *complete*. Show that a closed subspace Y of a complete metric space (X,d) is complete.

(b) Let (X,d) be a metric space. For non-empty $A \subseteq X$ and $r \geqslant 0$, define the r-*expansion* $E_r(A)$ by $E_r(A) = \bigcup_{a \in A} \overline{B}(a,r)$, where $\overline{B}(a,r)$ is the closed ball of radius r centred at a. Given non-empty $A, B \subseteq X$, is it always true that $B \subseteq E_r(A)$ if and only if $A \subseteq E_r(B)$? Justify your answer.

Let $H(X)$ denote the set of non-empty closed and bounded subsets of X. Given $A, B \in H(X)$, define

$$d_H(A,B) = \inf\{r \geqslant 0 : A \subseteq E_r(B) \text{ and } B \subseteq E_r(A)\}.$$

Show that d_H is a metric on $H(X)$. Would this continue to hold if the word 'closed' were omitted from the definition of $H(X)$? [You may assume that d_H is well defined.]

Show that the function $\theta : X \to H(X)$ defined by $x \mapsto \{x\}$ is a distance-preserving map, and that its image is closed in $H(X)$. Deduce that if $(H(X), d_H)$ is complete, so is (X,d).

Paper 2, Section II
10F Analysis and Topology

(a) Define what it means for a topological space (X,τ) to be *compact*. Define what it means for (X,τ) to be *Hausdorff*.

Show that a closed subspace Y of a compact space (X,τ) is compact.

Let (X,τ) be a compact Hausdorff space. Show that for any two disjoint closed subsets A and B of X, there exist disjoint open sets U and V containing A and B, respectively.

(b) A topological space (X,τ) is called *locally compact* if for each $x \in X$ and every neighbourhood U of x, U contains a compact neighbourhood K of x. Show that a compact Hausdorff space is locally compact.

Let (X,τ) be a locally compact Hausdorff space. Let $A \subseteq X$ be such that $A \cap K$ is closed in K for every compact $K \subseteq X$. Show that A is closed.

Paper 3, Section II

11F Analysis and Topology

Let X, Y be non-empty sets.

(a) Let d_X, d_Y be metrics on X, Y, respectively. Define what it means for a function $f : X \to Y$ to be *uniformly continuous.*

We say that a sequence of functions $f_n : X \to Y$ *converges uniformly* to a function $f : X \to Y$ *with respect to* d_Y if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ and for all $x \in X$, $d_Y(f_n(x), f(x)) < \epsilon$.

Show that a uniform limit of uniformly continuous functions $f_n : X \to Y$ is uniformly continuous. Give an example to show that the conclusion is false if convergence is pointwise but not uniform.

(b) Recall that two metrics d_1 and d_2 on X are *equivalent* if the identity map $\mathrm{id} : (X, d_1) \to (X, d_2)$ is a homeomorphism.

Show that if d_1 and d_2 are such that there exist constants $\alpha, \beta > 0$ with the property that for every $x, y \in X$,

$$\alpha d_1(x, y) \leqslant d_2(x, y) \leqslant \beta d_1(x, y),$$

then d_1 and d_2 are equivalent.

Does the reverse conclusion hold? Give a proof or a counterexample as appropriate.

If d_3 and d_4 are equivalent metrics on Y, is it true that a sequence of functions $f_n : X \to Y$ converges uniformly with respect to d_3 if and only if it converges uniformly with respect to d_4? Give a proof or a counterexample as appropriate.

Paper 4, Section II

10F Analysis and Topology

What does it mean for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be *differentiable* at $x \in \mathbb{R}^2$? Define the *derivative* $Df|_x$ and the *partial derivatives* $D_1 f(x)$ and $D_2 f(x)$ of f at $x \in \mathbb{R}^2$.

Show that if the partial derivatives of f exist in some open ball around $x \in \mathbb{R}^2$ and are continuous at x, then f is differentiable at x.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Find the partial derivatives of f at every point in $\mathbb{R}^2$. Are $D_1 f$ and $D_2 f$ continuous at $(0, 0)$? Is f differentiable at $(0, 0)$? Justify your answers.

Is it true that if f is differentiable everywhere in $\mathbb{R}^2$ then in a neighbourhood of each point at least one of the partial derivatives is bounded? Give a proof or a counterexample as appropriate.

Paper 4, Section I

3F Complex Analysis

Define what it means for $f : U \to \mathbb{C}$ to be *holomorphic* on a domain U.

State Morera's theorem.

Deduce that the function f defined on $\mathbb{C}$ by

$$f(z) = \int_0^1 \frac{e^{tz}}{1+t^2} dt$$

is holomorphic.

Give an example to show that a holomorphic function need not possess an anti-derivative on its domain.

[*Any further results you use should be stated clearly.*]

Paper 3, Section II

13F Complex Analysis

Define the *winding number* of a closed curve $\gamma : [a, b] \to \mathbb{C}$ about a point $w \in \mathbb{C}$ which is not in the image of γ. [You do not need to justify its existence.]

State the argument principle on a domain U bounded by a closed curve γ.

Deduce Rouché's theorem, which you should state carefully.

Let f be non-constant and holomorphic on an open set containing the closed unit disc $\overline{D}$. Suppose that $|f(z)| \geqslant 1$ for all z satisfying $|z| = 1$, and that there exists z_0 in the unit disc D such that $|f(z_0)| < 1$. Show that the image of f contains D.

Let g be holomorphic and non-zero on the punctured unit disc $D^* = D \setminus \{0\}$ such that g'/g has a simple pole at 0. Show that there exists a non-zero integer k such that h'/h has a removable singularity at 0, where h is defined by $h(z) = z^{-k} g(z)$.

Paper 1, Section I

3B Complex Analysis OR Complex Methods

Let $f(z) = \cosh \pi z$. Show that $z \mapsto \zeta = f(z)$ defines a mapping that is conformal from the complex z-plane to the complex ζ-plane, except at certain critical points which you should identify. Find the image in the ζ-plane of the strip

$$S = \{x + iy : 0 < x < \infty,\ 0 < y < 1\},$$

identifying clearly the image of each of the three line segments

$$L_1 = \{x + iy : 0 < x < \infty,\ y = 0\}, \qquad L_2 = \{x + iy : x = 0,\ 0 < y < 1\}$$

and

$$L_3 = \{x + iy : 0 < x < \infty,\ y = 1\}.$$

Paper 1, Section II

12F Complex Analysis OR Complex Methods

(a) Define what it means for a holomorphic function on a domain $U \setminus \{a\}$ to have (i) a *removable singularity*, (ii) a *pole of order* k, (iii) an *essential singularity* at $z = a$.

(b) Let f be holomorphic on the punctured unit disc $D^* = D \setminus \{0\}$ such that for all $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leqslant 1.$$

Show that f has a removable singularity at $z = 0$.

(c) Let $h(z) = \tan z$.

(i) Classify the singularities of $h(z)$ in $\mathbb{C}$.

(ii) Find the first two terms of the Laurent expansion of $h(1/z)$ around $z_k = \frac{2}{(2k+1)\pi}$, $k \in \mathbb{Z}$.

(iii) Classify the singularities of $\exp(h(1/z))$ in $\mathbb{C}$.

Paper 2, Section II

12B Complex Analysis OR Complex Methods

By considering the integral of an appropriate function on a semi-circular contour in the upper half plane, or otherwise, compute

$$\int_0^\infty \frac{(\ln x)^4}{1 + x^2}\, dx.$$

[*Hint: You may use that* $\displaystyle\int_0^\infty \frac{(\ln x)^2}{1 + x^2}\, dx = \frac{\pi^3}{8}$.]

Paper 3, Section I

3B Complex Methods

State Cauchy's theorem.

Calculate the Fourier transform of the function

$$f_{a,b}(x) = \exp[-ax^2 + ibx],$$

where $a > 0$ and b are real numbers, making sure to justify any change of variables you use.

[*Take the Fourier transform* $\hat{f}$ *of a function* $f : \mathbb{R} \to \mathbb{C}$ *to be given by* $\hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx$.]

Paper 4, Section II

12B Complex Methods

(i) Calculate the Laplace transform of the function defined for $0 \leqslant t < \infty$ by $f(t) = H(t - t_0)$ where H is the Heaviside function defined by $H(t) = 1$ if $t \geqslant 0$ and $H(t) = 0$ otherwise. (Here t_0 is an arbitrary positive number.)

(ii) Use the Fourier transform and contour integration to find the Green function defined by

$$-\frac{d^2G}{dx^2} + m^2 G = \delta(x), \qquad G(x) \to 0 \text{ as } |x| \to \infty,$$

where $m > 0$ and $-\infty < x < +\infty$. Explain why this Green function makes sense for $m \in \mathbb{C}$ with positive real part, and use it to write down a solution to

$$-\frac{d^2u}{dx^2} + m^2 u = f(x), \qquad u(x) \to 0 \text{ as } |x| \to \infty.$$

[*Take the Fourier transform* $\hat{G}$ *of* G *to be given by* $\hat{G}(k) = \int_{-\infty}^{+\infty} e^{-ikx} G(x) dx$.]

(iii) Use the Laplace transform to obtain an integral expression for the solution $u = u(t, x)$ of the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for } -\infty < x < +\infty, \quad 0 \leqslant t < +\infty$$

$$u(0, x) = 0\,, \quad u_t(0, x) = f(x)\,.$$

[*You may assume that* $u(t, x)$ *and* $f(x)$ *vanish for* $|x|$ *sufficiently large.*]

Paper 2, Section I

4C Electromagnetism

The two equations of magnetostatics are

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Explain briefly how the current density $\mathbf{J}$ can be non-zero even though the charge density vanishes.

Explain how a vector potential $\mathbf{A}$ can be introduced to solve one of these equations. Is $\mathbf{A}$ unique?

Show that in Cartesian coordinates (x, y, z) the following current density is consistent with charge conservation:

$$\mathbf{J} = J_0 \begin{pmatrix} \sin(\lambda z) \\ \cos(\lambda z) \\ 0 \end{pmatrix}$$

with λ and J_0 constant. What is the resulting magnetic field? What is the vector potential?

[*Hint: Consider* $\nabla \times \mathbf{J}$.]

Paper 4, Section I

5C Electromagnetism

State Faraday's law of induction, defining any terms that appear in the equation.

A circular wire loop has resistance R and lies in the $z = 0$ plane in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ with $B > 0$. The radius of the loop varies in time as $r(t)$. What is the current in the wire?

Distinguishing the two situations $\dot{r}(t) > 0$ and $\dot{r}(t) < 0$, draw a picture showing the magnetic field due to the induced current. Is the magnetic field increased or decreased inside the loop? In what direction is the Lorentz force on the wire in each case?

Paper 1, Section II

15C Electromagnetism

The vector potential $\mathbf{A}$ is related to the steady current density $\mathbf{J}$ by

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{J_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, .$$

Show that this vector potential obeys $\nabla \cdot \mathbf{A} = 0$, stating clearly any assumption that you make.

Show that, far from a localised current, the vector potential can be written as

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{r^3} + \ldots$$

where $r = |\mathbf{x}|$ and $\mathbf{m}$ is the magnetic dipole moment, which you should define in terms of $\mathbf{J}$.

What are the dimensions of $\mathbf{J}$ and of $\mathbf{m}$? Compute the magnetic dipole $\mathbf{m}$ for:

(i) a circular thin wire, with charge per unit length η and radius R, rotating around the axis of symmetry $\hat{\mathbf{n}}$ that is normal to the plane of the hoop, with angular velocity ω;

(ii) a circular disc with charge per unit area σ and radius R, rotating around the axis of symmetry $\hat{\mathbf{n}}$ that is normal to the plane of the disc, with angular velocity ω.

Paper 2, Section II

16C Electromagnetism

Throughout this question, use the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

The electromagnetic covector potential is

$$A_\mu = (-\Phi/c, A_i),$$

where Φ is the electrostatic potential, A_i are the components of the vector potential and c is the speed of light. The field-strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where ∂_μ denotes differentiation with respect to the spacetime coordinates $x^\mu = (ct, x^i)$.

Define a gauge transformation of A_μ and show that it leaves $F_{\mu\nu}$ unchanged.

Compute the components of $F_{\mu\nu}$ in terms of the electric and magnetic fields, which you should define in terms of Φ and A_i.

Explain how two of the four Maxwell equations follow automatically from the definition of $F_{\mu\nu}$. Show how the remaining two Maxwell equations follow from

$$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu$$

for some 4-vector j^μ that you should define.

Consider the tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right) .$$

Compute T^{00} in terms of the electric and magnetic fields and identify its physical meaning.

Show that, for any vector k_μ that is null (i.e. lightlike), if $T^{\mu\nu}k_\mu k_\nu$ is non-vanishing then it has a particular sign that you should determine. [*Hint: Consider a particular inertial frame chosen to simplify your expression.*]

Paper 3, Section II

15C Electromagnetism

Explain why all points of a conductor must sit on an equipotential. What does this imply for the electric field near the surface of a conductor? Derive an expression for the surface charge of a conductor in terms of the electric field.

Two spherical conducting shells, with radii R_1 and $R_2 > R_1$ are connected by a long, thin conducting rod of length $d \gg R_1 + R_2$. A charge Q is deposited on the spheres. What fraction of this charge resides on each shell? [*You may ignore the effect of the electric field from one shell on the other, and neglect the charge on the rod.*]

The same two spherical shells are are now placed concentrically around the origin. Again, they are connected by a thin conducting rod. What fraction of the charge Q sits on each shell? [*Again, you may neglect the charge on the rod.*]

A neutral conducting sphere of radius R is placed in a constant electric field $\mathbf{E} = E\hat{\mathbf{z}}$. Work in spherical polar coordinates, with $z = r\cos\theta$, and find a solution for the potential Φ outside the sphere of the form

$$\Phi = \alpha\left(r + \frac{\beta}{r^2}\right)\cos\theta$$

for some α and β that you should determine. What is the induced surface charge on the sphere? Confirm that the sphere is indeed neutral.

Paper 2, Section I

5D Fluid Dynamics

A fluid has velocity $\mathbf{u} = (y, ax)$ in Cartesian coordinates (x, y), where a is a real constant. Show that the flow is incompressible, determine a stream function $\psi(x, y)$ for the flow, and sketch the streamlines for $a > 0$ and for $a < 0$.

For what value of a is the flow also irrotational? In this case, determine a velocity potential $\phi(x, y)$ for the flow.

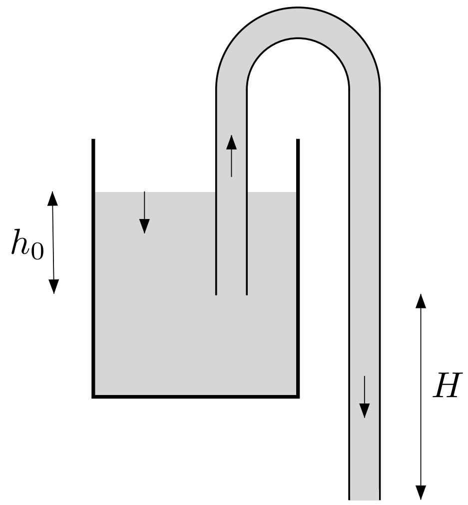
Paper 3, Section I

7D Fluid Dynamics

Starting from Euler's equations for steady, inviscid flow $\mathbf{u}$ of an incompressible fluid of uniform density ρ, subject to a body force $-\nabla\chi$, prove that $\mathbf{u} \cdot \nabla H = 0$, where $H \equiv \frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi$ and p is the fluid pressure. Interpret this equation physically.
[*You may use the identity* $\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u} \cdot \nabla \mathbf{u}$.]

Fluid initially occupies a tank with uniform horizontal cross-sectional area A. It is syphoned out of the tank using a tube of cross-sectional area $a \ll A$ as shown in the diagram below, in which flow directions (not magnitudes) are indicated. One end of the tube is held at a distance h_0 below the initial position of the free surface of the fluid in the tank, while the other end is held at a distance H below that. The tube is full of fluid and drains freely from its lower end into the surrounding air. Assuming the flow to be quasi-steady, show that the fluid level in the tank reaches the upper end of the tube after a time

$$t = \sqrt{2}\left(\frac{A^2}{a^2} - 1\right)^{1/2} \frac{\sqrt{H + h_0} - \sqrt{H}}{\sqrt{g}}.$$



Paper 1, Section II

16D Fluid Dynamics

A layer of fluid of uniform thickness h, density ρ and dynamic viscosity μ flows steadily down a rigid plane that is inclined at angle α to the horizontal. The surrounding air has uniform pressure p_0 but blows upslope, exerting a uniform shear stress τ on the fluid surface.

Write down the equations and boundary conditions describing parallel viscous flow in the fluid layer. Solve these to determine the pressure and velocity fields in the fluid. Hence, determine the surface velocity u_h, the shear stress τ_0 exerted by the fluid on the slope and the volume flux of fluid q per unit width across the plane. Determine how large a shear stress the blowing air must exert to cause (i) the surface velocity to be upslope (ii) the stress on the plane to be upslope (iii) the volume flux to be upslope, and order these measures of flow reversal by the magnitude of shear stress required.

Paper 3, Section II

16D Fluid Dynamics

A solid sphere of radius a moves in a straight line with speed U through fluid of density ρ that is at rest far from the sphere. Calculate the velocity potential ϕ for inviscid, irrotational flow of the surrounding fluid. Calculate the velocity components in the frame of reference in which the fluid is at rest far from the sphere. Hence calculate the total kinetic energy of the fluid.

Now suppose that the sphere has density $\rho_s > \rho$ and falls with speed U under gravity g. Write down the rate of change of the potential energy of the system (sphere plus fluid). By considering the rate of change of the total energy of the system (potential plus kinetic) or otherwise, show that

$$\frac{dU}{dt} = \frac{\rho_s - \rho}{\rho_s + \frac{1}{2}\rho} g.$$

Paper 4, Section II

16D Fluid Dynamics

A thin, horizontal layer of fluid of height $h = h_0 + \eta(x, y, t)$ flows with horizontal velocity components $\mathbf{u} = (u, v, 0)$ relative to a rotating frame of reference with Coriolis parameter $\mathbf{f} = (0, 0, f)$, in which (x, y, z) are Cartesian coordinates and where h_0 and f are constant and u and v are independent of z. When $\eta \ll h_0$, $\mathbf{u}$ and η satisfy the linearised equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta,$$

$$\frac{\partial \eta}{\partial t} + h_0 \nabla \cdot \mathbf{u} = 0,$$

where g is the acceleration due to gravity and $\nabla \equiv (\partial/\partial x, \partial/\partial y, 0)$ is the horizontal gradient operator.

Show that the linearised potential vorticity $\boldsymbol{\omega} - (\eta/h_0)\mathbf{f}$ is independent of time, where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the relative vorticity.

Suppose that $\eta = \eta_0$ when $\mathbf{u} \equiv 0$. Derive the evolution equation

$$\frac{\partial^2 \eta}{\partial t^2} - g h_0 \nabla^2 \eta + f^2 \eta = f^2 \eta_0.$$

Given that the fluid starts at rest with

$$\eta_0 = \begin{cases} \epsilon, & |x| < a \\ 0, & |x| > a \end{cases}$$

where ϵ is constant, determine the steady state $\eta_\infty(x)$ to which the system settles. Draw a sketch of the corresponding velocity field.

Paper 1, Section I

2E Geometry

Suppose a closed orientable surface Σ of genus g is obtained by identifying pairs of edges of a $2n$-gon. By considering Euler characteristics, show that $n \geqslant 2g$.

Draw a regular hyperbolic octagon in the Poincaré disc model, indicating edge identifications to produce a genus 2 surface. What are the interior angles of the octagon?

Paper 3, Section I

2G Geometry

Given a smooth function $h : \mathbb{R}^2 \to \mathbb{R}$, show that the graph

$$\Gamma = \{(x, y, z) : z = h(x, y)\}$$

is a smooth surface in $\mathbb{R}^3$. Write down a parametrisation of Γ, and compute the first fundamental form and Gauss map with respect to it.

Paper 1, Section II

11E Geometry

Let $\Sigma \subset \mathbb{R}^3$ be an embedded smooth surface and $\gamma : I \to \mathbb{R}^3$ be a smooth curve contained in Σ, where $I \subset \mathbb{R}$ is an open interval.

(i) Define what it means for γ to be a *geodesic* in Σ, and what it means for a geodesic to be *complete*.

Now suppose Σ_1 and Σ_2 are embedded smooth surfaces in $\mathbb{R}^3$ that are tangent to each other along the image of a curve $\gamma : I \to \mathbb{R}^3$.

(ii) Show that γ is a geodesic in Σ_1 if and only if it is a geodesic in Σ_2.

(iii) Let Σ_1 be the surface parametrised by

$$\sigma(u, v) = \big((2 + \cos v)\cos u, (2 + \cos v)\sin u, u + \sin v\big).$$

Sketch Σ_1 and, by considering a suitable surface Σ_2, show that the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(t) = (\cos t, \sin t, t)$$

is a geodesic in Σ_1. [You should justify why γ is a geodesic in Σ_2.]

(iv) Show that Σ_1 contains a complete geodesic Γ that is disjoint from this γ.

Paper 2, Section II

11G Geometry

The torus T^2 and Klein bottle K can both be described as quotients of $\mathbb{R}^2$ by equivalence relations $\sim$ and $\simeq$ given by

$$(x, y) \sim (x + a, y + b) \quad \text{for} \quad (a, b) \in \mathbb{Z}^2$$

and

$$(x, y) \simeq (x + c, (-1)^c y + d) \quad \text{for} \quad (c, d) \in \mathbb{Z}^2\ ,$$

respectively. Equip T^2 and K with the standard flat Riemannian metrics induced from $\mathbb{R}^2$ by these quotient constructions.

(a) Show that the map $\widetilde{\pi} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\widetilde{\pi}(x, y) = (2x, y)$ induces a well-defined 2:1 continuous map $\pi : T^2 \to K$.

(b) Draw a fundamental domain, with arrows indicating boundary gluing directions, for each of the two quotients.

(c) On separate copies of the fundamental domain for K, draw the images of closed geodesics γ_1, γ_2 and γ_3 with the following properties: γ_1 cuts K into a Möbius strip; γ_2 cuts K into a cylinder; γ_3 intersects itself in a single point.

(d) For each i, how many closed geodesics $\widetilde{\gamma}_i$ are contained in the preimage of the image of γ_i under π? For the purpose of counting, we consider two closed geodesics the same if they have the same image. On separate copies of the fundamental domain for T^2, draw examples of $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$.

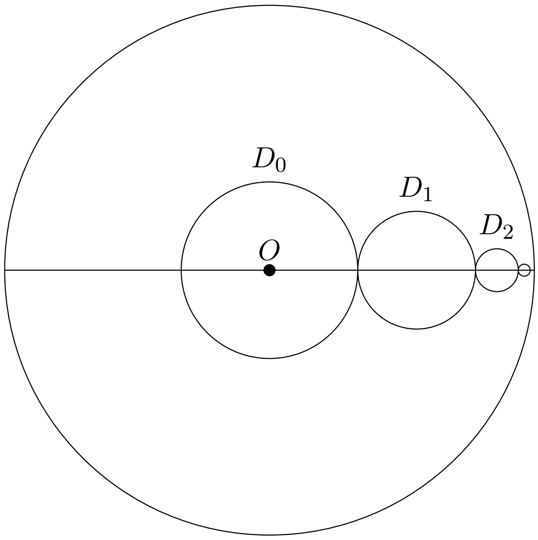
Paper 3, Section II

12E Geometry

Consider the Poincaré disc model of the hyperbolic plane, with Riemannian metric

$$\frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - x^2 - y^2)^2}.$$

Let $D_0, \ldots, D_n$ be distinct closed hyperbolic discs of hyperbolic area $\frac{\pi}{2}$, such that D_0 is centred at the origin O and each successive D_i is centred on the positive real axis and tangent to D_{i-1}, as shown in the figure below. Show that the hyperbolic centre of D_n is at $(4^n - 1)/(4^n + 1)$. [If you use any formulae for hyperbolic lengths or areas then they should be proved.]



Now consider the upper-half-plane model. Let $D'_0, \ldots, D'_n$ be distinct closed hyperbolic discs of hyperbolic area $\pi/2$ such that D'_0 has hyperbolic centre at $(0, 1)$ and each successive D'_i has hyperbolic centre on the line $y = 1$ and is tangent to D'_{i-1}. For $n \geqslant 3$, is there an isometry from the Poincaré disc model to the upper-half-plane model which carries each D_i to D'_i? Briefly justify your answer.

Paper 4, Section II

11G Geometry

Fix real numbers $a \in (0,1]$ and $b \in (0, \sqrt{2a - a^2})$. Let $f : [-4,4] \to \mathbb{R}$ be an even function that is smooth and strictly positive on $(-4,4)$ with the properties that

$f(x) = \sqrt{1-(3-x)^2}$ for $x \geqslant 2+a$

$f(x) = b$ for $x \in [-2+a, 2-a]$

f'' has a unique zero in $(2-a, 2+a)$.

Let $\Sigma \subset \mathbb{R}^3$ be the smooth surface defined by

$$x \in [-4,4] \quad \text{and} \quad f(x)^2 = y^2 + z^2.$$

(a) Sketch Σ in $\mathbb{R}^3$. Sketch its orthogonal projection onto the (x,z)-plane, and mark on this diagram (without proof) the regions of Σ where its Gaussian curvature K is positive, negative and zero respectively.

(b) Compute the integral of K over the region $R \subset \Sigma$ where $x \in [2-a, 2+a]$. [*You may use without proof the fact that a spherical disc of spherical radius θ has area $2\pi(1-\cos\theta)$.*]

(c) Show that the polygons obtained by cutting R along $y = 0$ are geodesic polygons only if $a = 1$.

Paper 2, Section I

1E Groups, Rings and Modules

State Eisenstein's irreducibility criterion.

(i) Let $n > 1$ be an integer. Prove that $X^{n-1} + X^{n-2} + \cdots + X + 1$ is irreducible in $\mathbb{Z}[X]$ if and only if n is a prime number.

(ii) Show that the polynomial $X^2 + Y^2 - 1$ in $\mathbb{Q}[X, Y]$ is irreducible. Would your argument work over any field?

Paper 3, Section I

1E Groups, Rings and Modules

(i) Suppose that A is a matrix over $\mathbb{Z}$. What is the *Smith normal form* for A? State the structure theorem for finitely-generated modules over $\mathbb{Z}$.

(ii) Find the Smith normal form of the matrix $\begin{pmatrix} -4 & -6 \\ 2 & 2 \end{pmatrix}$. Justify your answer.

Suppose that M is the $\mathbb{Z}$-module with generators e_1, e_2, subject to the relations $-4e_1 + 2e_2 = -6e_1 + 2e_2 = 0$. Describe M in terms of the structure theorem.

(iii) An abelian group is called *indecomposable* if it cannot be written as the direct sum of two non-trivial subgroups. Show that a finite group is indecomposable if and only if it is cyclic of prime power order.

Paper 1, Section II

9E Groups, Rings and Modules

(a) State Sylow's theorems.

(i) Identify the Sylow 2-subgroups and the Sylow 3-subgroups in the symmetric group S_3.

(ii) Identify the Sylow 2-subgroups of S_4.

(iii) Identify the Sylow 2-subgroups of the alternating group A_5.

(b) Let G be a finite group that has no subgroup of index 2. Let P be a Sylow 2-subgroup of G, let H be a subgroup of index 2 in P, and let x be an element of order 2 in G.

(i) Show that x acts as an even permutation on the set of cosets of H in G. Deduce that x must fix some points of this set.

(ii) Deduce that x must be conjugate to some element of H.

Paper 2, Section II

9E Groups, Rings and Modules

(a) If R is a Noetherian ring, show that R/I is Noetherian for each ideal I in R.

State the Hilbert basis theorem.

Explain briefly why $\mathbb{Z}$ is Noetherian. Deduce from these results that the ring $\mathbb{Z}[\sqrt{d}]$ for a non-square integer d is Noetherian.

(b) Let K be any field. Consider the set

$$R = \Big\{ f(X,Y) = \sum_{i,j} c_{ij} X^i Y^j \in K[X,Y] : c_{0j} = c_{j0} = 0 \text{ whenever } j > 0 \Big\}.$$

Verify that R is a subring of $K[X,Y]$ and determine, with justification, whether or not R is Noetherian.

Paper 3, Section II

10E Groups, Rings and Modules

(a) (i) Let R be a commutative unital ring. Show that an ideal I of R is prime if and only if R/I is an integral domain.

(ii) R is said to be *Boolean* if $r^2 = r$ for all $r \in R$. If R is Boolean, prove that $r + r = 0$ for all $r \in R$. Show also that if R is a non-zero integral domain and is Boolean, it is isomorphic to the field of two elements. Deduce that in a Boolean ring every prime ideal is maximal.

(b) (i) Let R be a commutative unital ring and let $R[X]$ be the ring of polynomials in X, with coefficients in R. Let I be an ideal of R and let $I[X]$ be the ideal of $R[X]$ consisting of all polynomials with coefficients in I. [*You may assume $I[X]$ is indeed an ideal.*] Show that $R[X]/I[X] \cong (R/I)[X]$. Deduce that if I is a prime ideal of R then $I[X]$ is a prime ideal of $R[X]$.

(ii) Give an example to show that if I is a maximal ideal of R then $I[X]$ need not be a maximal ideal of $R[X]$.

(c) In this part, we assume the prime number p is odd.

Let $\mathbb{F}_p$ be the field of p elements. Prove that its multiplicative group $\mathbb{F}_p^\times = \mathbb{F}_p \setminus \{0\}$ is a cyclic group.

Consider the group homomorphism $\phi : \mathbb{F}_p^\times \to \mathbb{F}_p^\times$ given by $x \mapsto x^2$, and let H be its image. Show that H has index 2 in $\mathbb{F}_p^\times$ and deduce that one of 2, 3 or 6 is a square in $\mathbb{F}_p$. Deduce that the polynomial $f(x) = x^6 - 11x^4 + 36x^2 - 36$ has a root modulo p.

Paper 4, Section II

9E Groups, Rings and Modules

Let R be a commutative unital ring.

(a) Let M be an R-module. What does it mean for M to be *free*? Assuming R is non-zero, if $R^n \cong R^m$ as R-modules, show that $n = m$.

If P and Q are R-modules such that $P \oplus Q$ is free, must P be free? Justify your answer.

(b) (i) We say that an R-module P is *projective* if, whenever we have R-module homomorphisms $f : M \to N$ and $g : P \to N$ with f surjective, then there exists a homomorphism $h : P \to M$ with $f \circ h = g$. Show that any free module (over an arbitrary commutative unital ring) is projective.

(ii) Suppose now that R is a principal ideal domain. Prove that any submodule N of a finitely-generated free module M over R is free. [*Hint: If N is a submodule of R^n for some n, you may wish to consider the composition of maps $N \to R^n \to R$, where the first map is inclusion and the second map is projection onto the first summand.*]

Deduce that a finitely-generated projective module over a principal ideal domain is free.

Paper 1, Section I

1G Linear Algebra

Define the *rank* and *nullity* of a linear map. State and prove the Rank-Nullity Theorem.

Let

$$W = \left\{ (x_i)_{i=1}^5 \in \mathbb{R}^5 : x_1 - x_2 + 3x_4 + 5x_5 = x_1 + x_2 + 6x_3 + 4x_4 - 2x_5 = x_1 + 3x_3 + 5x_4 - 4x_5 = 0 \right\} .$$

Find $\dim W$ and a basis for W.

Paper 4, Section I

1G Linear Algebra

State a theorem classifying $n \times n$ complex matrices up to similarity.

Let α be an endomorphism of an n-dimensional complex vector space. Define the *algebraic multiplicity* a_λ and the *geometric multiplicity* g_λ of an eigenvalue λ of α. Express a_λ and g_λ as well as the minimal polynomial of α in terms of a representation of α using the classification above.

Let α be represented by the 3×3 matrix

$$A = \begin{pmatrix} 5 & 0 & 3 \\ -1 & -1 & -1 \\ -6 & 0 & -4 \end{pmatrix} .$$

Find the eigenvalues of α and their algebraic and geometric multiplicities. Find the minimal polynomial of α.

Paper 1, Section II
8G Linear Algebra

(a) Define the *dual space* V^* of a vector space V over a field F. Show that if V is finite-dimensional then so is V^*, and that $\dim V^* = \dim V$.

Let $U \leqslant V$. Define the *annihilator* U° of U. Provide an expression, with proof, for $\dim U^\circ$ in terms of $\dim U$ in the case when V is finite-dimensional. Deduce that if $U \neq V$ then there exists $f \in U^\circ$ such that $f \neq 0$.

Let $\alpha\colon V \to W$ be a linear map between finite-dimensional vector spaces over F. Define the *dual map* $\alpha^*\colon W^* \to V^*$. Prove that $\ker \alpha^* = (\operatorname{im}\alpha)^\circ$ and $\operatorname{im}\alpha^* = (\ker\alpha)^\circ$.

Let V be a finite-dimensional vector space over F and $U \leqslant V$. By considering the quotient map $Q\colon V \to V/U$ and the inclusion map $J\colon U \to V$, show that $\big(V/U\big)^*$ is isomorphic to U°, and that U^* is isomorphic to V^*/U°.

(b) Let V be a vector space over a field F. Let $q_1, \ldots, q_n \in V^*$ be linearly independent. Show that the linear map $Q\colon V \to F^n$ given by $Q(x) = \big(q_j(x)\big)_{j=1}^n$ is surjective. Deduce that if $f \in V^*$ and $\bigcap_{j=1}^n \ker q_j \subseteq \ker f$ then f is in the span of $q_1, \ldots, q_n$.

Paper 2, Section II
8G Linear Algebra

(a) Let A be an $n \times n$ complex matrix. Define the *characteristic polynomial* of A. Show that A is similar to an upper-triangular matrix.

Define the *minimal polynomial* m_A of A. Prove that m_A exists and is unique. Prove that $\deg(m_A) \leqslant n$.

[*You may assume properties of determinants and results about matrix representation of linear maps. Any other results used must be proved.*]

(b) Let V be the real vector space of all real-valued functions on $\mathbb{R}$. For each $r \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, define $D_r\colon V \to V$ by $(D_r f)(x) = f(x+r) - f(x)$ for $f \in V$, $x \in \mathbb{R}$. Find the eigenvalues of D_r and show that the corresponding eigenspaces are infinite-dimensional. Show further that $D_r D_s = D_s D_r$ for all $r, s \in \mathbb{R}^\times$.

Call $f \in V$ *periodic* if $f \in \ker D_r$ for some $r \in \mathbb{R}^\times$. Show that a polynomial function in V of degree n cannot be written as a sum of n periodic functions.

Paper 3, Section II

9G Linear Algebra

Let V be a finite-dimensional complex inner product space with inner product $\langle \cdot, \cdot \rangle$. A map $f \colon V \to \mathbb{C}$ is *conjugate-linear* if $f(\lambda v + \mu w) = \bar{\lambda} f(v) + \bar{\mu} f(w)$ for all $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$. A map $\theta \colon V \times V \to \mathbb{C}$ is a *sesquilinear form* if the map $v \mapsto \theta(v, w)$ is linear for each $w \in V$ and the map $w \mapsto \theta(v, w)$ is conjugate-linear for each $v \in V$. Show that for each such θ, there is a unique map $\beta \colon V \to V$ such that $\theta(x, y) = \langle x, \beta(y) \rangle$ for all $x, y \in V$, and moreover that β is linear.

Let $\alpha \in \mathrm{End}(V)$. Use the results above to show the existence and uniqueness of the adjoint α^* of α. Prove the following statements. [*Standard properties of adjoints can be assumed.*]

(a) For a subspace U of V, $\alpha(U) \subseteq U$ if and only if $\alpha^*(U^\perp) \subseteq U^\perp$.

(b) If $\langle \alpha(x), x \rangle = 0$ for all $x \in V$, then $\alpha = 0$. Does the same hold in a real inner product space? Justify your answer.

(c) $\alpha\alpha^* = \alpha^*\alpha$ if and only if $\|\alpha(x)\| = \|\alpha^*(x)\|$ for all $x \in V$. Does the same hold in a real inner product space? Justify your answer.

(d) If $\alpha\alpha^* = \alpha^*\alpha$, then there is an orthonormal basis of V consisting of eigenvectors of α.

Paper 4, Section II

8G Linear Algebra

(a) Let $m, n \in \mathbb{N}$. Show that two $m \times n$ matrices A and A' over a field F are equivalent if and only if there exist vector spaces V, W, a linear map $\alpha\colon V \to W$ and bases B, B' of V and C, C' of W such that $A = [\alpha]_{B,C}$ and $A' = [\alpha]_{B',C'}$. [*You may assume the correspondence between composition of linear maps and products of matrices.*]

Define the *column rank* and the *row rank* of an $m \times n$ matrix A over F and prove that they are equal. [*You may assume the Rank-Nullity Theorem. Other results used should be proved.*]

(b) Fix $m, n \in \mathbb{N}$. Let $[m] = \{1, \ldots, m\}$ and $[n] = \{1, \ldots, n\}$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$. For $x = (x_j)_{j=1}^n \in \mathbb{C}^n$, let $\mathrm{supp}(x) = \{j \in [n] : x_j \neq 0\}$, and for $B \subseteq [n]$, let Bx be the vector in $\mathbb{C}^n$ with jth coordinate x_j if $j \in B$ and 0 if $j \notin B$.

Let $v_1, \ldots, v_m$ be linearly independent vectors in $\mathbb{C}^n$. Show that there is an injection $f\colon [m] \to [n]$ such that $f(i) \in \mathrm{supp}(v_i)$ for all $i \in [m]$. [*Hint: You may use the following result. If $F\colon [m] \to \mathcal{P}[n]$, where $\mathcal{P}[n]$ is the power set of $[n]$, satisfies*

$$|A| \leqslant \left| \bigcup_{i \in A} F(i) \right|$$

for all $A \subseteq [m]$ then there is an injection $f\colon [m] \to [n]$ such that $f(i) \in F(i)$ for all $i \in [m]$.]

Using part (a), or otherwise, show that there is a subset B of $[n]$ of size m such that $Bv_1, \ldots, Bv_m$ are linearly independent.

Deduce that there is an injection $f\colon [m] \to [n]$ such that $f(i) \in \mathrm{supp}(v_i)$ for all $i \in [m]$, and that

$$\big(\{e_j : j \in [n]\} \setminus \{e_{f(i)} : i \in [m]\}\big) \cup \{v_i : i \in [m]\}$$

is a basis of $\mathbb{C}^n$.

Paper 3, Section I

8H Markov Chains

Let $\{X_n : n \geqslant 1\}$ be independent, identically distributed, integer-valued random variables. Define

(i) $S_n = \sum_{i=1}^{n} X_i$

(ii) $L_n = \min\{X_1, X_2, \ldots, X_n\}$

(iii) $K_n = X_n + X_{n-1}$, with $X_0 = 0$.

Which of the sequences $\{X_n\}, \{S_n\}, \{L_n\}, \{K_n\}$ are necessarily Markov chains? Justify your answers.

Paper 4, Section I

7H Markov Chains

A taxi driver moves between the airport A and two hotels B and C according to the following rules: if she is at the airport, she will proceed to one of the hotels with equal probability; if she is at a hotel, she will return to the airport with probability $\frac{3}{4}$ and travel to the other hotel with probability $\frac{1}{4}$.

(a) What is the transition matrix for the corresponding Markov chain?

(b) Suppose the driver begins at the airport at time 0.

(i) Find the probability for each of her three possible locations at time 2.

(ii) What is the probability that the driver is at the airport at time $n \geqslant 1$?

Paper 1, Section II

19H Markov Chains

Suppose $\{X_n\}_{n \geqslant 0}$ is a Markov chain such that there exists a pair (i, j) of distinct states that are "symmetric" in the sense that

$$\mathbb{P}(T_j < T_i \mid X_0 = i) = \mathbb{P}(T_i < T_j \mid X_0 = j),$$

where $T_i = \min\{n \geqslant 1 : X_n = i\}$. Denote this common conditional probability by α. Suppose $X_0 = i$, and let N denote the number of visits to j before the chain revisits i.

(a) Compute $\mathbb{E}[N]$.

(b) For each $k \geqslant 0$, compute $\mathbb{P}(N = k)$ as a function of α.

(c) Prove that if an irreducible Markov chain has an invariant distribution π, two states i and j are symmetric if and only if $\pi(i) = \pi(j)$.

[*Standard results can be quoted without proof.*]

Paper 2, Section II

18H Markov Chains

An urn initially contains m green balls and $m+2$ red balls. A ball is picked at random: if it is green, a red ball is also removed and both are discarded; if it is red, it is replaced together with an extra red and an extra green ball. This is repeated until there are no green balls in the urn. Compute the probability that the process terminates. [Your answer should be a function of m.]

[*Standard results can be quoted without proof, provided they are stated clearly.*]

Paper 2, Section I

3B Methods

For integer n, the Chebychev polynomials T_n satisfy the equation

$$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0\,, \qquad -1 < x < 1\,.$$

Put this equation into Sturm-Liouville form and derive an orthogonality relation between T_n and T_m for $n \neq m$. Find a second order differential equation satisfied by the derivatives $U_n = T_n'$, and an orthogonality relation between U_n and U_m for $n \neq m$.

Paper 3, Section I

5B Methods

Let $u(r,\theta)$ satisfy the Laplace equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in the annulus $\mathcal{A}$ given in polar coordinates by

$$\mathcal{A} = \{(r,\theta):\ a < r < b\,, 0 \leqslant \theta < 2\pi\}\,.$$

Use separation of variables to derive a general expression for u. Given boundary conditions $u(a,\theta) = 0$ and $u(b,\theta) = \cos 2\theta$, find u explicitly.

Paper 1, Section II

13B Methods

The Green function $G(x,\xi)$ satisfies

$$G'' + \alpha(x)G' + \beta(x)G = \delta(x-\xi) \quad \text{for} \quad 0 < x < 1,$$

with $G'(0,\xi) = G'(1,\xi) = 0$, where primes denote differentiation with respect to x.

Find the function $c(\xi)$ for $0 < \xi < 1$ such that the Green function can be written as

$$G(x,\xi) = \begin{cases} c(\xi)y_1(x)y_2(\xi) & \text{for } \ 0 < x < \xi \\ c(\xi)y_2(x)y_1(\xi) & \text{for } \ \xi < x < 1 \end{cases}$$

in terms of linearly independent solutions $y_1(x)$ and $y_2(x)$ of

$$y'' + \alpha(x)y' + \beta(x)y = 0 \quad \text{for} \quad 0 < x < 1$$

that satisfy $y_1'(0) = 0$ and $y_2'(1) = 0$.

Deduce that if $\alpha(x) = 0$ for all x then $G(x,\xi) = G(\xi,x)$.

Find G explicitly for the case $\alpha(x) = 0$ and $\beta(x) = -1$ for all x. Hence solve the equation $y'' - y = x$ on the interval $[0,1]$ with boundary conditions $y'(0) = 0 = y'(1)$.

Paper 2, Section II

14B Methods

Define the *convolution* $f * g$ of two functions on the real line. The function F is defined by

$$F(x) = \begin{cases} e^{-x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0. \end{cases}$$

A sequence of functions $F_1, F_2, F_3 \ldots$ is defined by $F_1 = F$, $F_n = F * F_{n-1}$ for $n \geqslant 2$ (so F_n is the n-fold convolution of F with itself). Use induction to find F_n, without using the Fourier transform.

The Fourier transform $\hat{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$\hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx.$$

Compute the Fourier transform $\hat{F}_n$.

State and prove the convolution theorem. Verify that $\hat{F}_n = (\hat{F})^n$.

Using the identity

$$\int_{-\infty}^{+\infty} e^{-ikx} dk = 2\pi\delta(x)$$

and interchanging the order of integration, show that the convolution theorem with an appropriate choice of g implies the Parseval identity

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(k)|^2 dk\,.$$

[*You may assume that f and $\hat{f}$ are integrable and decrease rapidly at infinity, and that the order of integration in multiple integrals can be interchanged.*]

Deduce the value of the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{(1+k^2)^{n+1}} dk$.

Paper 3, Section II

14B Methods

Transverse oscillations $y = y(t, x)$ of a string in a resisting medium are governed by the damped wave equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial t} = 0\,, \quad 0 < x < 1\,, \quad t > 0\,.$$

Assuming the string is fixed at $x = 0$ and $x = 1$ so that $y(t, 0) = 0 = y(t, 1)$, use separation of variables to derive an expression for the solution with given initial values

$$y(0, x) = \sum_{n=1}^{\infty} a_n \sin n\pi x\,, \qquad \frac{\partial y}{\partial t}(0, x) = 0\,. \tag{$*$}$$

Calculate the Fourier coefficients $\{a_n\}$ in the particular case

$$y(0, x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \leqslant x \leqslant 1\,. \end{cases}$$

For $(*)$ in the case of general coefficients $\{a_n\}$, use the Parseval identity to calculate the energy

$$E(t) = \frac{1}{2} \int_0^1 \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 dx$$

at time t in terms of the coefficients $\{a_n\}$. Hence, or otherwise, show that the energy is decreasing.

Paper 4, Section II

14B Methods

Let a and $\kappa \geqslant 0$ be real constants. Consider the problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \kappa\frac{\partial^2 u}{\partial x^2}$$

with initial condition $u(0,x) = u_0(x)$, where $u_0(x)$ is a given function. [*You may assume* u_0 *and* u *to be smooth and decreasing to zero as* $|x| \to \infty$ *as needed.*]

(i) For $a = 0, \kappa > 0$ write down an integral expression for the solution in terms of the function

$$K_t(x) = \begin{cases} (4\pi\kappa t)^{-\frac{1}{2}} \exp\left[-\frac{x^2}{4\kappa t}\right] & \text{if } t > 0 \\ 0 & \text{if } t \leqslant 0\,. \end{cases}$$

Explain briefly why your formula for $u(t,x)$ reduces to $u_0(x)$ when t tends to zero by considering the behaviour of K_t in this limit, and give a sketch to illustrate.

(ii) For $\kappa = 0$, use the method of characteristics to find the solution.

(iii) For the general case with $\kappa > 0$ and $a \in \mathbb{R}$ arbitrary, find an integral expression for the solution.

Paper 1, Section I

5A Numerical Analysis

Let p_n be the real polynomial of degree at most n that interpolates a continuous function $f(x)$ at the $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$. Define the *divided difference* $f[x_0, x_1, \ldots, x_k]$.

Starting from the Lagrange formula, prove that p_n satisfies

$$p_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i),$$

the Newton form. [You need not write an explicit expression for the divided difference.]

Write down (without proof) the recurrence relation for the divided difference. For $n = 2$, draw a diagram that explains how the divided difference $f[x_0, x_1, \ldots, x_k]$ could be computed efficiently. Find the exact number of divisions needed for such computations with $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$.

Paper 4, Section I

6D Numerical Analysis

The composite, mid-point, quadrature rule for computing the integral $I(f) = \displaystyle\int_0^1 f(x)\,dx$ is given by

$$I_N(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \quad \text{with} \quad x_n = \frac{1}{N}\left(n + \tfrac{1}{2}\right).$$

Determine the order of convergence $I_N(f) \to I(f)$ of this scheme as $N \to \infty$ if f is at least twice differentiable on [0, 1].

A different function $f(x)$ is known to have a square-root singularity at $x = 0$, so that $f(x) = x^{-1/2} g(x)$, where $g(x)$ is analytic on [0,1]. Determine, with justification, a sequence y_n such that the quadrature

$$J_N(f) = \frac{2}{N} \sum_{n=0}^{N-1} y_n^{1/2} f(y_n)$$

has the same order of convergence $J_N(f) \to I(f)$ as the scheme above. [*Hint: Consider a change of variables in* $I(f)$.]

Paper 1, Section II

17A Numerical Analysis

Suppose that the real orthogonal matrix $\Omega^{[p,q]} \in \mathbb{R}^{m\times m}$ with $1 \leqslant p < q \leqslant m$ is a Givens rotation with rotation angle θ. Write down the form of $\Omega^{[p,q]}$.

Show that for any matrix $A \in \mathbb{R}^{m\times m}$ and for any given j it is possible to choose θ such that the matrix $\Omega^{[p,q]}A$ satisfies $(\Omega^{[p,q]}A)_{qj}{=}0$, where $1 \leqslant j \leqslant m$.

Let

$$A = \begin{pmatrix} \sqrt{2} & 0 & \frac{1-\sqrt{3}}{\sqrt{2}} \\ 0 & \sqrt{3} & 1 \\ \sqrt{2} & \sqrt{2} & \frac{1+\sqrt{3}}{\sqrt{2}} \end{pmatrix}.$$

By applying the product $\Omega^{[p,q]}\Omega^{[p',q']}$ of two Givens rotations (picking appropriate values for p, p', q and q', with $p' < p$), find a factorisation of the matrix $A \in \mathbb{R}^{3\times 3}$ of the form $A = QR$, where $Q \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix, $R \in \mathbb{R}^{3\times 3}$ is an upper triangular matrix for which the leading non-zero element in each row is positive, and every entry of Q and R is written as a rational number multiplied by the square root of an integer.

Paper 2, Section II

17D Numerical Analysis

(a) Define the *linear stability domain* of a numerical method to solve the system of equations

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}).$$

What does it mean for the numerical method to be *A-stable*? Determine the linear stability domains for the forward and backward Euler methods, and deduce whether each is A-stable.

(b) What does it mean for a differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ to be *stiff*? Illustrate your answer with the example

$$\mathbf{y}' = \mathsf{M}\mathbf{y}, \qquad \mathsf{M} = \begin{pmatrix} -10 & 10 \\ 81 & -91 \end{pmatrix},$$

determining what step size h is required to maintain stability of the forward and backward Euler methods, respectively, applied to this ODE.

(c) For the ODE $\mathbf{y}' = \mathsf{M}\mathbf{y}$ with a general matrix M, use the Milne device with the forward and backward Euler methods to determine a local error estimate, and describe how you would use that estimate to construct a stable integration scheme that achieves a desired tolerance in a reasonably small number of steps.

Paper 3, Section II

17A Numerical Analysis

For a function $f \in C^4[-1,2]$ consider the following approximation of $f''(-1)$:

$$f''(-1) \approx \eta(f) = a_{-1} f(-1) + a_0 f(0) + a_1 f(1) + a_2 f(2),$$

with the error

$$e(f) = f''(-1) - \eta(f).$$

We want to find the smallest constant c such that

$$|e(f)| \leqslant c \max_{x \in [-1,2]} |f''''(x)|,$$

where $f''''(x)$ is the fourth derivative of f.

State the necessary conditions on the approximation scheme η for the inequality above to be valid with some $c < \infty$. Hence determine the coefficients a_{-1}, a_0, a_1, a_2.

State the Peano kernel theorem.

Assuming that the Peano kernel is non-negative in $x \in [-1,2]$, use the Peano kernel theorem to find the smallest constant c.

Paper 1, Section I

7H Optimisation

(a) Find the dual problem of the following linear program:

$$\begin{aligned} \text{minimise} \quad & \sum_{i=1}^{n} a_i x_i \\ \text{subject to:} \quad & \sum_{i=1}^{n} b_i x_i \geqslant t \\ & \sum_{i=1}^{n} c_i x_i \geqslant r \\ & x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n, \end{aligned}$$

where $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n, \{c_i\}_{i=1}^n, t$ and r are arbitrary real numbers.

(b) Using part (a) or otherwise, solve the following optimisation problem:

$$\begin{aligned} \text{minimise} \quad & 3x_1 + 2x_2 + 2x_3 - 2x_4 \\ \text{subject to:} \quad & x_1 + x_2 - x_4 \geqslant 7 \\ & x_1 + x_3 - 2x_4 \geqslant 11 \\ & x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant 4. \end{aligned}$$

You should find the minimum value of the objective function and the vector (x_1, x_2, x_3, x_4) that attains this minimum value.

Paper 2, Section I

7H Optimisation

Consider the following optimisation problem:

$$\begin{aligned} \text{minimise} \quad & x_1 \log x_1 - x_2 \\ \text{subject to:} \quad & x_1 + x_2 \leqslant c \\ & \sqrt{x_2} \leqslant d \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

At $x_1 = 0$, the value of $x_1 \log x_1$ is defined to be equal to 0, its limiting value.

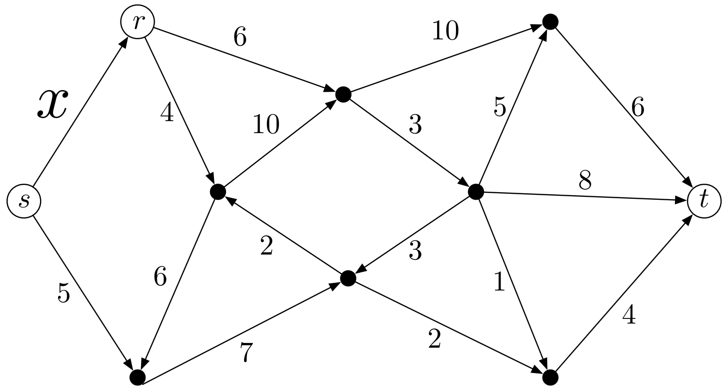
(a) Use the Lagrange method to search for a solution when $c = 3/e^2$ and $d = 2/e$, where e is the base of the natural logarithm.

(b) Now use the Lagrange method to search for a solution when $c = 3/e^2$ and $d = 1/e$. Explain your observations.

Paper 3, Section II
19H Optimisation

(a) For $n > 1$, consider a directed graph G with vertices $V = \{1, 2, \ldots, n\}$ and edge set E. Following the usual convention, if $(i, j) \in E$, then G has a directed edge from vertex i to vertex j. Each edge $(i, j) \in E$ has an associated capacity $C_{ij} \geqslant 0$. Vertex 1 is designated as the source and vertex n is designated as the sink. State and prove the max-flow min-cut theorem.

(b) Consider the directed graph with edge capacities as shown in the figure below, with source s, sink t, and an intermediate vertex r:



(i) Let the capacity of the edge from s to r be x. Find the maximum flow from s to t when $x = 4$. Verify your answer by identifying a cut whose capacity equals your answer.

(ii) Let $\delta^*(x)$ be the maximum flow from s to t, as a function of x. Derive a formula for $\delta^*(x)$ when $x \geqslant 0$.

Paper 4, Section II

18H Optimisation

(a) Describe Newton's method for minimising a function $f : \mathbb{R}^d \to \mathbb{R}$. Denote by x^* a minimiser of f, and by x_k the kth iterate in Newton's method. Stating clearly any assumptions f must satisfy, provide an upper bound on $f(x_k) - f(x^*)$.

(b) Suppose $a \geqslant 1$. Consider the following algorithm used by the ancient Babylonians to approximate $\sqrt{a}$: set $x_0 \geqslant 1$ and, for each $k \geqslant 0$, iteratively define

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right).$$

Prove that all the iterates lie in $[1, \infty)$. Derive the algorithm above as a consequence of applying Newton's method for minimising a suitable function $f : [1, \infty) \to \mathbb{R}$.

(c) For a given $a \geqslant 1$, identify a range of values for x_0 such that $x_k \to \sqrt{a}$ as $k \to \infty$. Derive an upper bound on $|x_k - \sqrt{a}|$. [*Hint: You may find the result in part (a) useful, as well as the equation* $x^3 - 3ax + 2a^{3/2} = (x - \sqrt{a})^2(x + 2\sqrt{a})$.]

Paper 3, Section I

6A Quantum Mechanics

Consider a one-dimensional system described by a wavefunction ψ for which $\langle \hat{p} \rangle = 0$ and $\langle \hat{x} \rangle = 0$.

(i) Write down the commutation relation between $\hat{x}$ and $\hat{p}$.

(ii) Define the *uncertainty* ΔO of an observable $\hat{O}$ in terms of $\langle \hat{O} \rangle$ and $\langle \hat{O}^2 \rangle$.

(iii) Considering the one-parameter family of states defined by

$$\psi_s(x) = (\hat{p} - is\hat{x})\psi(x),$$

where $s \in \mathbb{R}$, derive the Heisenberg uncertainty relation between Δx and Δp.

Paper 4, Section I

4A Quantum Mechanics

Write down the time-independent Schrödinger equation for a particle of mass m with wavefunction $\psi(x)$ moving in a potential $V(x)$.

Consider the one-dimensional potential $V(x) = -V_0$ for $|x| < a$ and $V(x) = 0$ for $|x| > a$, for constant $V_0 > 0$.

By integrating the Schrödinger equation over a small interval around $x = a$, analyse the continuity of $\psi(x)$ and $\psi'(x)$ at $x = a$.

Show that

$$\psi(x) = \begin{cases} A\exp(-\eta|x|) & \text{for } |x| > a, \\ B\cos(kx) & \text{for } |x| < a, \end{cases}$$

is a solution of the time-independent Schrödinger equation, deriving two necessary relationships between η and k in the process.

Draw a diagram in the (k, η) plane that indicates the locus of the lowest energy level when $k < \pi/(2a)$.

Paper 1, Section II

14A Quantum Mechanics

Muonium is an atom consisting of an electron of mass m_e and charge e in the potential of an anti-muon (of opposite charge to the electron) of mass $m_\mu \gg m_e$. You may assume that the anti-muon is long-lived enough to form a bound state and that it produces the same force field as a proton.

(i) In what ways (if any) should the quantum mechanical description of the electron in muonium differ to that in the Hydrogen atom? Give reasons for your answer.

(ii) For fixed orbital angular momentum quantum number l, write down the equation satisfied by the radial part of the electron wavefunction $R(r)$. Show that it has solutions of the form

$$R(r) \propto r^l \exp\left(-\frac{r}{a(l+1)}\right),$$

where a is a constant that you should determine. Find the energy.

(iii) Atoms of muonium consistent with such solutions are prepared with energy

$$E = -\frac{e^2}{72\pi\epsilon_0 a}.$$

Show that the average of measurements of the electron-anti-muon spatial separation in a large set of such muonium atoms is equal to ta, where t is a number that you should find.

(iv) Taking one of the prepared muonium atoms with energy E, what is the numerical probability that an immediate measurement of the total orbital angular momentum yields $\hbar$?

[*Hint: the time-independent Schrödinger equation of the Hydrogen atom is*

$$-\frac{\hbar^2}{2m_e r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{2m_e r^2}\hat{L}^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi.$$

The normalised energy eigenstates of the Hydrogen atom have the form

$$\psi_{lm}(r,\theta,\phi) = R(r)Y_{lm}(\theta,\phi),$$

where Y_{lm} are orbital angular momentum eigenstates satisfying

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1)Y_{lm}, \qquad \hat{L}_3 Y_{lm} = \hbar m Y_{lm},$$

where $l = 1,\ 2, \ldots$ and $m = 0,\ \pm 1,\ \pm 2,\ \ldots, \pm l$.

You may assume that $\int_0^\infty dt\, t^l e^{-t} = l!$ and that $\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\ |Y_{lm}|^2 = 1$.]

Paper 2, Section II

15A Quantum Mechanics

(i) Calculate the commutation relation between a position operator $\hat{x}$ and its associated momentum $\hat{p}_x = -i\hbar\partial/\partial x$.

(ii) Write down the time-dependent Schrödinger equation for a quantum mechanical system with Hamiltonian $\hat{H}$ and wavefunction ψ.

(iii) Calculate the rate of change of the expectation value of some operator $\hat{O} = \hat{O}(t)$ in terms of $\langle[\hat{O},\ \hat{H}]\rangle$ and $\langle\frac{\partial\hat{O}}{\partial t}\rangle$.

(iv) Express each of $[\hat{x},\ \hat{p}_x^2]$ and $[\hat{x}^2,\ \hat{p}_x]$ in terms of a single operator.

(v) Consider the two-dimensional harmonic oscillator whose Hamiltonian is

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2).$$

Setting $\hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$, calculate $[\hat{L},\ \hat{H}]$.

(vi) Changing variables to $z = x + iy$, consider the two degenerate energy eigenstates

$$\psi = Az\exp(-\beta|z|^2) \qquad \text{and} \qquad \psi^* = Az^*\exp(-\beta|z|^2)$$

where A and β are positive real constants. At time $t = 0$, a state $\frac{\sqrt{5}}{3}\psi + \frac{2}{3}\psi^*$ is prepared. What is the expectation value of $\hat{L}$ at a later time $t > 0$?

Paper 4, Section II

15A Quantum Mechanics

A quantum mechanical particle moves in an inverted harmonic oscillator potential. Its wavefunction $\psi(x,t)$ evolves according to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}x^2\psi.$$

(i) Show that there exists a solution of the form

$$\psi(x,t) = A(t)\exp(-B(t)x^2)$$

provided that

$$\frac{dA}{dt} = -i\hbar AB$$

and

$$\frac{dB}{dt} = -\frac{i}{2\hbar} - 2i\hbar B^2.$$

(ii) Show that $B = \xi\tan(\phi + \alpha t)$ solves the equation for B, where ξ and α are constants that you should find and ϕ is a constant of integration.

(iii) Find $A(t)$ in terms of $\cos(\phi+\alpha t)$. You need not calculate its normalisation explicitly.

(iv) Compute the expectation values of $\hat{x}^2$ and $\hat{p}^2$ as functions of B.

[*Hint: You may use* $\dfrac{\int_{-\infty}^{\infty} dx\ e^{-Cx^2}x^2}{\int_{-\infty}^{\infty} dx\ e^{-Cx^2}} = \dfrac{1}{2C}$.]

Paper 1, Section I
6H Statistics

(a) Define what it means for a statistic to be *sufficient*. State the factorization criterion.

(b) What does it mean for a sufficient statistic to be *minimal sufficient*?

Now suppose X is a single sample from a $N(0, \sigma^2)$ distribution, where σ is the parameter we wish to estimate.

(c) Prove that $|X|$ is a sufficient statistic. Is $|X|$ minimal sufficient?

(d) Suppose we instead have i.i.d. samples $X_1, X_2, \ldots, X_n \sim N(0, \sigma^2)$, where $n \geqslant 2$. Is $\sum_{i=1}^{n} |X_i|$ a sufficient statistic for estimating σ?

[*Standard results can be quoted without proof.*]

Paper 2, Section I
6H Statistics

After losing a large amount of money, an unlucky gambler questions whether the game was fair and the die was really unbiased. The last 90 rolls of this die gave the following results:

score on the die	1	2	3	4	5	6
number of times it occurred	20	15	12	17	9	17

(i) Suppose the gambler wishes to test the hypothesis that the die is fair. What are the null and alternative hypotheses?

(ii) Describe Pearson's test. What is the limiting distribution of the Pearson statistic under the null hypothesis?

(iii) Compute the Pearson statistic for this test.

(iv) What is the asymptotic p-value of the test (written as a quantile of an appropriate distribution)?

[*Standard results can be quoted without proof, provided they are stated clearly.*]

Paper 1, Section II

18H Statistics

Suppose X_1 and X_2 are i.i.d. from a Uniform$(\theta, \theta+1)$ distribution. For testing

$$H_0 : \theta \leqslant 0 \quad \text{against} \quad H_1 : \theta > 0,$$

consider the following two tests:

Test 1: Reject H_0 if $X_1 > 0.95$;
Test 2: Reject H_0 if $X_1 + X_2 > C$.

(a) Derive a formula for the probability density function of $X_1 + X_2$ and plot the function.

(b) Find the value of C so that Test 2 has the same size as Test 1.

(c) Calculate the power of each test as a function of θ.

(d) Is either Test 1 or Test 2 uniformly most powerful for testing the hypotheses? Justify your answer.

Paper 3, Section II

18H Statistics

Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from a $N(\mu, 1)$ population, and we wish to test the hypothesis

$$H_0 : \mu = 0 \quad \text{against} \quad H_1 : \mu \neq 0.$$

(a) Define a level-α hypothesis test based on $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and find an expression for the p-value $p(\bar{x})$ corresponding to a value $\bar{X} = \bar{x}$ in terms of the standard normal cumulative distribution function Φ.

Now consider a prior distribution on μ, defined as follows: with probability $\frac{1}{2}$, take $\mu = 0$, and with probability $\frac{1}{2}$, draw a value of μ from a $N(0, \tau^2)$ distribution, where τ is known.

(b) What is the conditional probability density function of $\bar{X}|\mu$? Derive an expression for the marginal probability density function $m(\bar{x})$ of $\bar{X}$ under the prior distribution described above.

[*Hint: The formula* $\int_{-\infty}^{\infty} e^{-ax^2-bx}dx = \sqrt{\frac{\pi}{a}}e^{b^2/(4a)}$, *for* $a > 0$, *may be helpful.*]

(c) Find an expression for the posterior probability $q(\bar{x})$ that H_0 is true, defined by $q(\bar{x}) = \mathbb{P}(\mu = 0|\bar{X} = \bar{x})$.

(d) Suppose we fix $n = 100$ and $\tau = 1$. Compare the probabilities $p(\bar{x})$ and $q(\bar{x})$ obtained in parts (a) and (c):

(i) Which probability is larger when $\bar{x}$ is close to 0?

(ii) Which probability is larger when $|\bar{x}|$ is large?

[*You may use the Taylor series approximation* $1 - \Phi(x) \approx \dfrac{e^{-x^2/2}}{\sqrt{2\pi}x}$, *which holds for large values of* x.]

[*Standard results can be quoted without proof.*]

Paper 4, Section II

17H Statistics

Observations $\{(x_i, Y_i)\}_{i=1}^n$ are made according to the model

$$Y_i = \alpha + \beta x_i + \epsilon_i,$$

where $\{x_i\}_{i=1}^n$ are fixed constants in $\mathbb{R}$ and $\epsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2)$, for a known value of σ.

(a) Derive expressions for maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ for α and β, respectively.

Now suppose the model is reparametrized as

$$Y_i = \alpha' + \beta'(x_i - \bar{x}) + \epsilon_i,$$

where $\bar{x} := \frac{1}{n}\sum_{i=1}^n x_i$. Let $\hat{\alpha}'$ and $\hat{\beta}'$ denote maximum likelihood estimators for α' and β', respectively.

(b) Show that $\hat{\beta}' = \hat{\beta}$.

(c) Show that in general, $\hat{\alpha}' \neq \hat{\alpha}$. In fact, show that $\hat{\alpha}' = \frac{1}{n}\sum_{i=1}^n Y_i$.

(d) What is the distribution of $\hat{\alpha}'$? Construct a 95% confidence interval for α' based on $\hat{\alpha}'$.

(e) Now suppose σ is unknown. Construct a 95% confidence interval for α' in this setting, and explain why it has the specified coverage.

[*Standard results can be quoted without proof.*]

Paper 1, Section I

4C Variational Principles

Describe how to use the method of Lagrange multipliers to extremise a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

You need to make a rectangular cardboard box. The bottom of the box must be three times thicker than the front and back. The sides of the box must be twice as thick as the front and back. The box has no top. The volume of the box must be equal to three. What are the lengths of the sides and height if you wish to minimise the amount of cardboard used?

Paper 3, Section I

4C Variational Principles

Given a Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}, t)$, what is the momentum $\mathbf{p}$ conjugate to $\mathbf{x}$? What is the Hamiltonian? Under what circumstances is energy conserved?

The dynamics of a particle with position $\mathbf{x} = (x, y, z)$ is governed by the Lagrangian

$$L = mc\sqrt{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}} - V(\mathbf{x}), \tag{1}$$

where m and c are positive constants, and $V(\mathbf{x})$ is a potential.

Determine the momentum $\mathbf{p}$ conjugate to $\mathbf{x}$. Determine the Euler-Lagrange equation.

Show that $\mathbf{p}\cdot\mathbf{p}$ is constant. Determine the Hamiltonian. Is it possible to reconstruct the Lagrangian from the Hamiltonian?

Paper 2, Section II

13C Variational Principles

A functional of $y(x)$ takes the form

$$I[y] = \int_0^{x_0} F(y', y, x)\, dx \ .$$

Derive the Euler-Lagrange equation and explain why solutions to this equation, assuming that they exist, extremise $I[y]$ under the assumption that $y(x)$ is fixed at each end.

The system is said to have *free boundary conditions* if $\partial F/\partial y' = 0$ at the end points. Explain why the solutions to the Euler-Lagrange equations, if they exist, also extremise $I[y]$ if free boundary conditions are imposed at each end.

Consider the functional of $y(x)$ and $z(x)$ given by

$$J[y, z] = \int_0^{x_0} \Big[y'^2 + z'^2 + 2yz \Big]\, dx \ .$$

Find the most general solution to the Euler-Lagrange equations subject to the requirement that $y(0) = z(0) = 0$. For which values of x_0 are there solutions if we impose free boundary conditions at x_0? Find these solutions.

Paper 4, Section II

13C Variational Principles

Three scalar fields, $\phi(\mathbf{x}, t)$, $\alpha(\mathbf{x}, t)$, and $\beta(\mathbf{x}, t)$, are each a function of the spatial coordinates $\mathbf{x} = (x_1, x_2, x_3)$ and time t. The dynamics of these fields is governed by extremising the functional

$$S[\phi, \alpha, \beta] = \int_{-\infty}^{+\infty} \Big[-\beta \frac{\partial \alpha}{\partial t} - \frac{1}{2} (\nabla\phi + \beta\nabla\alpha) \cdot (\nabla\phi + \beta\nabla\alpha) \ \Big]\, dt\, d^3x \ .$$

Write down the Euler-Lagrange equations for ϕ, α and β.

Define the vector field

$$\mathbf{u} = \nabla\phi + \beta\nabla\alpha \ .$$

Show that the Euler-Lagrange equations can be written as

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \frac{\partial \alpha}{\partial t} + \mathbf{u} \cdot \nabla\alpha = 0 \quad \text{and} \quad \frac{\partial \beta}{\partial t} + \mathbf{u} \cdot \nabla\beta = 0 \ .$$

Hence show that the vector field $\mathbf{u}$ obeys

$$\frac{\partial u_i}{\partial t} + \mathbf{u} \cdot \nabla u_i = -\frac{\partial p}{\partial x_i} \ ,$$

where $p = -\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + f(\dot\phi, \dot\alpha, \beta)$ and $f(\dot\phi, \dot\alpha, \beta)$ is a function that you should determine, and where $\dot\phi$ and $\dot\alpha$ are the partial derivatives of ϕ and α with respect to t.

END OF PAPER