UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday, 25 May, 2010 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

(a) Using Cartesian coordinates, show that

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}),$$

and hence that

[8] $$\boldsymbol{\nabla}\times((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}) = (\boldsymbol{\nabla}\cdot\mathbf{u})(\boldsymbol{\nabla}\times\mathbf{u}) + (\mathbf{u}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\times\mathbf{u}) - ((\boldsymbol{\nabla}\times\mathbf{u})\cdot\boldsymbol{\nabla})\mathbf{u}.$$

(b) Consider a vector field $\mathbf{v}(x, y, z)$, which may be expressed in Cartesian coordinates as

$$\mathbf{v} = \left(-\frac{\kappa y}{2\pi[x^2+y^2]}, \frac{\kappa x}{2\pi[x^2+y^2]}, 0\right).$$

[6] Show that $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$ everywhere except along the line $x = y = 0$.

Show that the line integral

$$\oint_C \mathbf{v}\cdot d\mathbf{r} \qquad (*)$$

is equal to zero for all curves $C = \partial S$ in the x-y plane which bound open surfaces S (also in the x-y plane) which do not intersect the line $x = y = 0$. What is the value of the
[6] integral (*) if the curve C bounds such a surface which **does** intersect this line?

[*You may find it useful to use cylindrical polar coordinates.*]

2C

Consider diffusion inside a circular tube (with very small cross-section) and circumference 2π. Let x denote the arc-length parameter $-\pi \leqslant x \leqslant \pi$, so that the density of the diffusing substance u satisfies (for $t > 0$)

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2},$$

with specified initial condition $u(x, 0) = f(x)$ for some function $f(x)$. What are the
[3] appropriate boundary conditions to impose on u at $x = \pm\pi$ for $t > 0$?

Use separation of variables to express $u(x, t)$ in terms of an appropriate infinite
[7] series.

Compute explicitly the coefficients of the above series in the case that $f(x) =$
[10] $(\pi - |x|)^2$, and identify the density distribution of the substance u as $t \to \infty$.

3C

Consider a linear differential operator L defined by

$$Ly = -\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y, \qquad 0 < x < +\infty.$$

By writing $y = z/x$ or otherwise, find those solutions of $Ly = 0$ which are either
[5] (i) bounded as $x \to 0$, or (ii) bounded as $x \to +\infty$.

Find the Green's function $G(x,\xi)$ satisfying

$$LG(x,\xi) = \delta(x-\xi),$$

[8] such that G is bounded as $x \to 0$ and G is bounded as $x \to +\infty$.

Use $G(x,\xi)$ to solve

$$Ly = \begin{cases} 1, & \text{for } 0 \leqslant x \leqslant R, \\ 0, & \text{for } \quad\quad x > R, \end{cases}$$

[7] with y bounded as $x \to 0$ and $x \to +\infty$.

[*It is convenient to consider the solution for $x > R$ and $x < R$ separately.*]

4C

Calculate the Fourier transform of the function

$$g(x) = e^{-\lambda|x|},$$

where λ is a positive constant, and hence or otherwise calculate the Fourier transform of the function

$$h(x) = \frac{1}{x^2 + \mu^2},$$

[8] where μ is a positive constant.

Consider Laplace's equation for $\psi(x, y)$ in the half-plane with prescribed boundary conditions at $y = 0$, i.e.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0; \qquad -\infty < x < \infty, \; y \geqslant 0,$$

where $\psi(x, 0) = f(x)$ is a known function with a well-defined Fourier transform, and where ψ tends to zero as $y \to \infty$, and $f(x) \to 0$ as $|x| \to \infty$.

By taking the Fourier transform with respect to x, and by applying the convolution theorem (which may be quoted without proof) show that

[8] $$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(x-u)^2 + y^2} \, du.$$

Find (in closed form) the solution when

[4] $$f(x) = \begin{cases} c = \text{constant}, & \text{for } |x| < a, \\ 0, & \text{otherwise}. \end{cases}$$

5B What is (i) an *eigenvalue*, and (ii) an *eigenvector*, of a complex $n \times n$ matrix A? Show that A has at least one eigenvector. [3]

Give an example of a non-diagonalizable $n \times n$ matrix (for some n). [2]

What is a *Hermitian* matrix? Explain briefly why a Hermitian matrix can always be diagonalized. [3]

In the remainder of this question A is a Hermitian matrix. Now assume that $\mathbf{e}_i$, for $i = 1, \ldots n$, is a complete set of eigenvectors for A, with corresponding eigenvalues λ_i. Prove that the eigenvalues λ_i are real. [1]

Assume from now on that all the eigenvalues are negative: $\lambda_i < 0$.

Obtain complete sets of eigenvectors and eigenvalues for A^{-1} and A^n for all $n = 1, 2 \ldots$ [4]

Prove that

$$A^{-1} = \int_0^\infty e^{tA} dt.$$

[7]

[*You may use without proof that any complex polynomial has a complex zero. If B is a matrix then $e^B = \sum_{n=0}^\infty B^n/(n!)$. If $B(t)$ is a matrix depending on t with entries $B_{ij}(t)$ then $\int_0^\infty B(t)dt$ means the matrix with entries $\int_0^\infty B_{ij}(t)dt$, when these integrals exist.*]

6B

(a) Give a real linear transformation $\mathbf{x} = L\mathbf{y}$ which converts the quadratic form $Q_1(\mathbf{x}) = x_1^2 + 4\,x_1x_2 + 5\,x_2^2 + 6\,x_3^2$ into $\widetilde{Q}_1(\mathbf{y}) = y_1^2 + y_2^2 + y_3^2$. What is the corresponding result for the quadratic form $Q_2(\mathbf{x}) = x_1^2 + 4\,x_1x_2 + 5\,x_2^2 - 6\,x_3^2$? [5]

(b) Define the trace $\mathrm{tr}(A)$ of a square matrix A and prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. For which complex numbers c do there exist $n \times n$ matrices A, B such that

$$AB - BA = cI\,,$$

where I is the identity matrix? For each complex number c either give an example or prove the non-existence of such matrices. [6]

(c) Let $A(\epsilon)$ be a symmetric $n \times n$ matrix for each real ϵ. The smallest eigenvalue of $A(\epsilon)$ is $\lambda(\epsilon)$, with corresponding real eigenvector $\mathbf{x} = \mathbf{x}(\epsilon)$ normalized so that $\mathbf{x}^T\mathbf{x} = 1$ for all ϵ, where ^{T} denotes transpose. Assuming that $A(\epsilon), \lambda(\epsilon), \mathbf{x}(\epsilon)$ vary smoothly with ϵ, show that

$$\left.\frac{d\lambda}{d\epsilon}\right|_{\epsilon=0} = \left.\mathbf{x}^T \frac{dA}{d\epsilon}\mathbf{x}\right|_{\epsilon=0},$$

where ^{T} denotes transpose. [9]

7A

Obtain the Cauchy-Riemann equations for the analytic function

[2] $$f(z) = u(x,y) + iv(x,y)\,.$$

Show that:

[2] (i) u and v satisfy Laplace's equation, $\nabla^2 u = \nabla^2 v = 0\,;$

[2] (ii) the level sets $u = \textit{constant}$ and $v = \textit{constant}$ are orthogonal, $\nabla u \cdot \nabla v = 0\,;$

[2] (iii) every stationary point of u is a stationary point of v and conversely;

[4] (iv) stationary points for which $\begin{vmatrix} \partial_{xx} u & \partial_{xy} u \\ \partial_{yx} u & \partial_{yy} u \end{vmatrix} \neq 0$ must be saddle points;

(v) if $f(z) = u(x,y) + iv(x,y)$ and $g(z) = s(x,y) + it(x,y)$ are analytic functions, then
[8] so is $g(f(z))$, and hence deduce that $s(u(x,y), v(x,y))$ satisfies Laplace's equation.

8C

Show that the origin is an ordinary point, and that $x = 1$ and $x = -1$ are regular singular points of the equation

$$(1 - x^2)\,\frac{d^2 y}{dx^2} - x\,\frac{dy}{dx} + p^2 y = 0\,, \qquad (*)$$

[4] where p is a real constant.

You may assume that there are two independent series solutions of the form

$$y_q(x) = x^q \sum_{n=0}^{\infty} a_n\, x^n\,, \qquad q = 0\,,1\,.$$

Find the recurrence relations for a_n for the two cases, and show that the series converge
[6] for $|x| < 1\,.$

Show that polynomial solutions $T_m(x)$ exist for $p = m$, where m is a non-negative integer. With the condition $T_m(1) = 1$, calculate all the coefficients for the cases
[6] $m = 0,\ 1,\ 2,\ 3\,.$

For $-1 \leqslant x \leqslant 1$ make the substitution $x = \cos\theta$, with $0 \leqslant \theta \leqslant \pi$, in the differential equation (*). Hence, or otherwise, show that $T_m(x) = \cos(m \cos^{-1} x)$ for any non-
[4] negative integer m.

9C

State the Euler equation obtained by making stationary

$$F[y] = \int_a^b f(x, y, y')\, dx\,,$$

with fixed values of $y(a)$ and $y(b)$, and show that if $f = f(y, y')$ is not an explicit function of x, then

$$y' \frac{\partial f}{\partial y'} - f = A\,,$$

[6] where A is a constant.

In an optical medium occupying the region $0 < y < h$, the speed of light is

$$c(y) = \frac{c_0}{(1 - ky)^{1/2}}\,, \qquad (0 < k < 1/h)\,.$$

[8] Show that the paths of light rays in the medium are parabolic.

Show also that, if a ray enters the medium at $(-x_0, 0)$ and leaves it at $(x_0, 0)$, then

$$(k\,x_0)^2 = 4\,k\,y_0(1 - k\,y_0)\,,$$

[6] where y_0 $(< h)$ is the greatest value of y attained on the ray path.

10C

Consider a Sturm-Liouville problem:

$$-\frac{d}{dx}\left(p(x)\,\frac{dy}{dx}\right) + q(x)\,y - \lambda w(x)\,y \;=\; 0\,,$$

defined for $a \leqslant x \leqslant b$, with $p > 0$ and $w > 0$ on the interval, and with boundary conditions $y(b) = y(a) = 0$. You may assume that this problem has a complete infinite set of orthonormal eigenfunctions y_i $(i = 0, 1,\, 2,\, \ldots)$ with associated (ordered) eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$.

Define a class of trial functions $y_{trial}(x)$ such that

$$y_{trial}(x) \;=\; A\,y_0(x) + A\sum_{i=1}^{\infty} c_i\, y_i\,, \qquad (\dagger)$$

for some non-zero constant A. Define

$$\Lambda[y] \;=\; \frac{\int_a^b (py'^{\,2} + qy^2)\,dx}{\int_a^b y^2 w\,dx} \;=\; \frac{F[y]}{G[y]}\,.$$

Show that

[4]
$$\lambda_{trial} \;\equiv\; \Lambda[y_{trial}] \;=\; \frac{\lambda_0 + \sum_{i=1}^{\infty} c_i^2\,\lambda_i}{1 + \sum_{i=1}^{\infty} c_i^2}\,. \qquad (*)$$

By taking variations of Λ, F and G explicitly, for general y satisfying boundary conditions of the above form, show that the stationary values of $\Lambda[y]$ are the eigenvalues λ_i, and hence deduce that $\Lambda[y]$ is bounded below by λ_0. (Euler's equation may be quoted
[6] without proof.)

Consider the specific problem

$$\frac{d^2y}{dx^2} + \lambda y \;=\; 0\,, \qquad 0 \leqslant x \leqslant 1\,,\;\; y(0) \;=\; y(1) \;=0\,.$$

Generate an estimate λ_{trial} for the smallest eigenvalue λ_0 by using the trial function
[4] $y_{trial} = x(1-x)$.

Represent $y_{trial} = x(1-x)$ as an infinite series of the form given in ($\dagger$) for this particular problem, and thus derive an expression for the ratio

[6]
$$\frac{c_1^2(\lambda_1 - \lambda_0)}{(\lambda_{trial} - \lambda_0)}\,.$$

END OF PAPER