UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 13 June, 2022 9:00 am to 12:00 pm

MATHEMATICS (1)

This is a closed book exam.

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed gold cover sheet.*

Tie up ***each*** *section B answer in a* ***separate*** *bundle, marked with the question number.* ***Do not join the bundles together****. For each bundle, a gold cover sheet* ***must*** *be completed and attached to the bundle.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your blind grade number and desk number, and should not include your name.

STATIONERY REQUIREMENTS
6 gold cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
No calculators may be used.
No electronic devices may be used.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) Factorise the expression $(a^3 - b^3)$ in terms of real-valued factors. [1]

(b) Simplify the expression $(a^{1/3} + b^{1/3})\left[a^{2/3} - (ab)^{1/3} + b^{2/3}\right]$. [1]

2 Solve for real x: $\sqrt{x^2 - 2x + 1} = 2$. [2]

3 Solve the inequality: $\frac{1}{x} \geqslant -1$. [2]

4 Solve the set of simultaneous equations for real x and y:

$$\begin{cases} 3^x + 3^y &= 2, \\ 3^{x+y} &= 1. \end{cases}$$

[2]

5 Evaluate

(a) $-2 + 4 - 6 + 8 - 10 + \ldots - 98 + 100$, [1]

(b) $\sin(\pi/4) - \sin^2(\pi/4) + \sin^3(\pi/4) - \sin^4(\pi/4) + \ldots$ [1]

6 Find the value of x at which the function $y = x^3 e^{-x}$ reaches its maximum in the range $0 \leqslant x < \infty$ and evaluate the value of y at this point. [2]

7 Sketch $y = \frac{\sin(x)}{x^2}$ for positive x and label the crossing points, if any, with the horizontal axis. [2]

8 Find the indefinite integral of $y = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$ for $x > 0$. [2]

9 Find the centre and the radius of the circle

$$x^2 - 4x + y^2 + 3 = 0.$$

[2]

10 In terms of parameter $a > 0$, what is the value of the dot product for vectors $(a, a+1)$ and $(a+1, a)$? Hence, what is the angle between these vectors if $a^2 + a = 1/2$? [2]

SECTION B

11Z

(a) Find all possible real and imaginary parts of the following expressions:

(i) $(i^i)^i$, [2]

(ii) $i^{(i^i)}$. [2]

(b) Describe, with the aid of a sketch, the curve in the Argand diagram whose equation is $|z+1+i| = 8$. [2]

(c) Describe, with the aid of sketches, the loci determined, for z on the curve in part (b), by the complex numbers

(i) $u = \frac{3}{2}z + \frac{1}{2}z^*$, [4]

(ii) $v = u + 4 + 3i$, [2]

(iii) $w = iv$. [2]

(d) Express $\sin 5\theta$ in terms of $\sin\theta$ and its powers, and find the values of θ such that $16\sin^5\theta = \sin 5\theta$ for $0 \leqslant \theta < 2\pi$. [6]

12X

(a) Consider $\int_{x=2y}^{2} \int_{y=0}^{1} x^2 y^2 dxdy$.

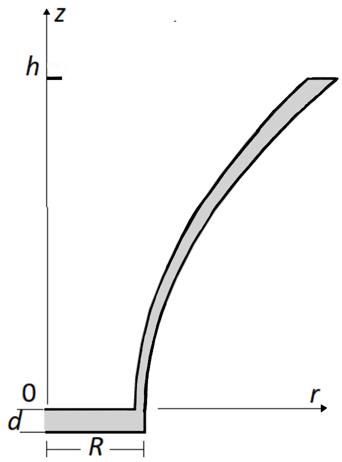
(i) Explain why the integration should be done over x first. [2]

(ii) Sketch the region of integration. [3]

(iii) Evaluate the integral. [3]

(iv) Change the order of integration and re-calculate the integral. [3]

(b) The radial ($r \geqslant 0$) cross section of a cup is shown in the diagram.



You may assume cylindrical symmetry about the z-axis. The outer curved line ($z \geqslant 0$) obeys $r = R(1+z^2/h^2)$. The inner curved line obeys $r = 0.9R(1+z^2/h^2)$. The cup is made from metal of density ρ.

Calculate the mass M of the cup. [9]

13Y

(a) By using an appropriate substitution and integrating factor, or otherwise, find in explicit form the general solution of the following equation,

$$\frac{dy}{dx} = -\frac{2x^2 + y^2 + x}{xy}.$$

[10]

(b) Solve the differential equation

$$\sin x \frac{dy}{dx} + 2y \cos x = 1,$$

subject to the boundary condition

$$y(\pi/2) = 1.$$

[10]

14W

(a) Consider a differential form $P(x, y)dx + Q(x, y)dy$ with

$$P(x, y) = y^2 \sin(ax) + xy^2 \cos(ax) \quad \text{and} \quad Q(x, y) = 2xy \sin(ax) \ ,$$

where a is a real, non-zero parameter.

(i) Find all values of a for which this differential form is exact and thus can be written as $df = P(x, y)dx + Q(x, y)dy$. [2]

(ii) Find $f(x, y)$. [4]

(b) Let the real function $f(u, v)$ be twice differentiable in both independent positive variables, $u = u(x, y)$ and $v = v(x, y)$, which depend on two other independent real variables, x and y, according to the following relations,

$$u(x, y) = 1 + x^2 + y^2 \quad \text{and} \quad v(x, y) = 1 + x^2 y^2 \ .$$

Find, in terms of x and y and partial derivatives of f with respect to u and v:

(i) $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, [2]

(ii) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$. [6]

For

$$f(u, v) = \ln(uv) \ , \quad \text{find} \quad \frac{\partial^2 f}{\partial x \partial y} \ ,$$

(iii) by using the expressions for the second derivative derived in (b)(ii), [3]

(iv) by first expressing f in terms of x and y and then finding the second derivative. [3]

15R

(a) Find the first three non-zero terms in the Taylor series expansion of the real-valued function $f(x) = \sqrt{a^2 + x^2}$ around $x = 0$, where the parameter a is any real number including zero. [7]

(b) Consider the real-valued function $f(x) = \ln[1 + t(x)]$ around $x = 0$, where the real-valued function $t(x)$ and all its derivatives exist in the sufficient neighbourhood of $x = 0$, and where $t(x)$ obeys the following properties: $1 + t(0) > 0$ and $t'''(0)t'(0) < 0$.

Find the first two non-zero terms in the Taylor series expansion of $f(x)$ about $x = 0$. Present your answers by considering separately the following cases;

(i) $t(0) \neq 0$, [4]

(ii) $t(0) = 0$ and $t''(0) \neq t'^2(0)$, [4]

(iii) $t(0) = 0$ and $t''(0) = t'^2(0)$, [5]

and explain how the assumed properties of $t(x)$ are used in your analysis.

16V

The continuous random variable, X has probability distribution

$$f(x) = \begin{cases} 0 & x < -2, \\ \frac{1+e^{-|x|}}{N} & -2 < x < 2, \\ 0 & x > 2, \end{cases}$$

where N is a constant.

(a) Find the normalisation factor, N. [4]

(b) Plot a graph of $f(x)$. [3]

(c) Find the expectation of X, $E[X]$. [2]

(d) Find the variance of the random variable X. [5]

(e) The continuous random variable, Y, has probability density

$$g(y) = \begin{cases} 0 & y < -2, \\ P(X \leqslant y)/M & -2 < y < 2, \\ 0 & y > 2, \end{cases}$$

where P denotes the probability. Find the normalisation factor, M. [6]

17T

(a) (i) Find relationships between

$$\int e^x \sin x \, \mathrm{d}x \quad \text{and} \quad \int e^x \cos x \, \mathrm{d}x,$$

[2]

(ii) and hence evaluate the integrals

$$\int e^x (\sin x - \cos x) \, \mathrm{d}x \quad \text{and} \quad \int e^x (\sin x + \cos x) \, \mathrm{d}x.$$

[2]

(b) By performing the integration, show that

$$\int e^{nx}(\sin x - \cos x) dx = -\frac{e^{nx}}{n^2+1}\left[(1+n)\cos x + (1-n)\sin x\right] + c,$$

where c is a constant. [7]

(c) Evaluate the definite integrals

(i)

$$\int_0^\infty x e^{-x^2} dx,$$

[2]

(ii)

$$I(n) = \int_0^\infty x^n e^{-x^2} dx,$$

in terms of integral $I(n-2)$, for integer $n \geqslant 2$. [5]

(iii) Hence determine $I(n)$ for $n = 2,\ 3,\ 4,\ 5$ and 6. [2]

[Hint: The integral $I(0) = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.]

18S

Let

$$\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

(a) Calculate $\text{Tr}(\boldsymbol{A})$, $\det(\boldsymbol{A})$, $\text{Tr}(\boldsymbol{A}^2)$, and $\det(\boldsymbol{A}^2)$. [4]

(b) Compute the eigenvalues and normalized eigenvectors of $\boldsymbol{A}$. [9]

(c) Find all solutions $\boldsymbol{x}$ of the matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{b} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix},$$

by expanding $\boldsymbol{x}$ and $\boldsymbol{b}$ as linear combinations of the eigenvectors found in (b), or by any other method.

Give a geometric interpretation of the solutions.

[7]

19V*

(a) Determine whether the following series converge or diverge. For those that converge, evaluate the series limit;

(i) $\sum_{n=1}^{\infty} \left[\frac{\sin((2n-1)\pi)}{n} - \frac{2\cos((2n-3)\pi)}{n} \right]$, [4]

(ii) $\sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{2^m}{3^m}$, [4]

(iii) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}\right)$, [4]

(iv) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. [4]

(b) Prove that the following series converges, and find the value it converges to;

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

[4]

20T*

(a) For general functions, $f(x)$, $g(x)$ and $h(x,t)$ write down the formula for

$$\frac{d}{dx}\int_{g(x)}^{f(x)} h(x,t)dt,$$

and hence evaluate the limit [2]

$$\lim_{x\to\infty}\frac{d}{dx}\int_{\sin(1/x)}^{\sqrt{x}} \frac{2t^4+1}{(t-2)(t^2+3)}dt.$$

[4]

(b) Consider the function

$$f(\alpha)=\int_0^{\infty}\frac{\alpha\ln(\alpha^2+x^2)}{1+x^2}dx,$$

where α is a real parameter. Demonstrate that

$$\frac{df}{d\alpha}-\frac{f}{\alpha}=\frac{\pi\alpha}{\alpha+1}.$$

[8]

Solve this equation for $f(\alpha)$. Hence show that

$$\int_0^{\infty}\frac{\alpha\ln(\alpha^2+x^2)}{1+x^2}dx=\pi\alpha\ln(\alpha+1).$$

[6]

END OF PAPER